PHY 505 - Hour Test 1
Solutions

1. See posted solutions to Homework No. 2 - Problem 1 on web site

2. A. Direct Computation by solving Poisson Equation

$$\phi(\vec{r}) = K \int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d^3r', \qquad K = \frac{1}{4\pi\epsilon_0}$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell=0}^{\infty} \frac{r_<^{\ell}}{r_>^{\ell+1}} P_\ell(\cos\theta)$$

i) Break up integration into 2 Regions $\begin{Bmatrix} r' < r \\ r' > r \end{Bmatrix}$ and use appropriate expansion for each region

ii) Since $\rho(r)$ is spherically symmetric, only $\ell=0$ contributes

Therefore, $$\phi(\vec{r}) = k \left\{ \frac{1}{r} \int_{r'<r} \rho(\vec{r}\,') d^3r' + \int_{r'>r} \frac{\rho(\vec{r}\,')}{r'} d^3r' \right.$$

$$\phi(\vec{r}) = \frac{kQ}{2r_0^3} \left\{ \frac{1}{r} \int_0^r e^{-r'/r_0} r'^2 dr' + \int_r^\infty e^{-r'/r_0} r' dr' \right\} \qquad (+)$$

In both integrals, let $x = r'/r_0$

$$\phi(\vec{r}) = \frac{kQ}{2}\left\{ \frac{1}{r}\int_0^{x_0} e^{-x} x^2\,dx + \frac{1}{r_0}\int_{x_0}^{\infty} e^{-x} x\,dx \right\} \qquad x_0 = r/r_0$$

By repeated integration by parts

$$\int_0^{x_0} e^{-x} x^2\,dx = -e^{-x}(x^2+2x+2)\Big|_0^{x_0}$$

$$= 2 - 2e^{-x_0}\left(1 + x_0 + \tfrac{1}{2}x_0^2\right)$$

$$\int_{x_0}^{\infty} e^{-x} x\,dx = e^{-x_0}(1+x_0)$$

Substituting in (7)

$$\phi(\vec{r}) = kQ\left(\frac{1}{r} - e^{-r/r_0}\left(\frac{1}{r} + \frac{1}{2r_0}\right)\right)$$

[Note: $\lim_{r\to\infty} \phi(\vec{r}) = \frac{kQ}{r} \to 0$

$\lim_{r\to 0} \phi(\vec{r}) = \frac{kQ}{2r_0}$]

B. Solution by Gauss Law

By spherical symmetry $\vec{E}(\vec{r}) = E(r)\,\hat{r}$

Then by Gauss Law

$$E(r) = \frac{k}{r^2} \int_{r'<r} \rho(\vec{r}')\,d^3r'$$

$$Q_{in} = \int_{r'<r} \rho(r')\,d^3r' = \frac{Q}{2r_0^3} \int_0^r e^{-r'/r_0}\, r'^2\,dr'$$

$$= Q\left[1 - e^{-r/r_0}\left(1 + \frac{r}{r_0} + \frac{r^2}{2r_0^2}\right)\right]$$

Then

$$\phi(\infty) - \phi(r) = -\int_r^\infty E(r')\,dr'$$

So

$$\phi(r) = kQ\left\{\int_{r_0}^\infty \frac{dr'}{r'^2}\right.$$

$$\left. - \frac{1}{r_0}\int_{x_0}^\infty e^{-x}\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)dx\right.$$

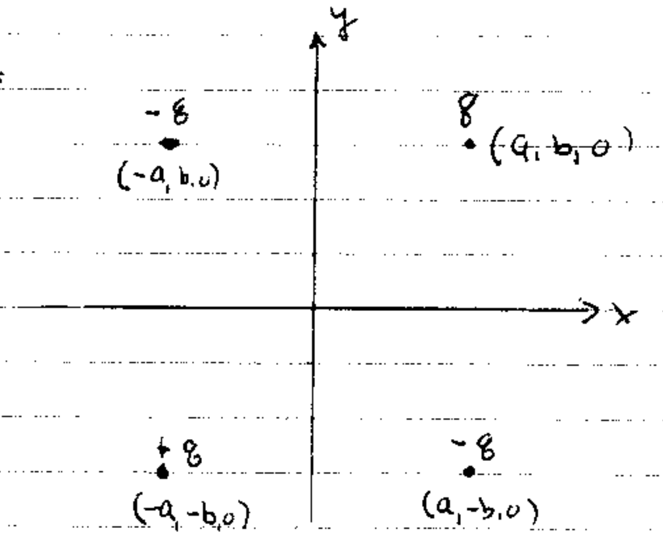
Note $\frac{d}{dx}\frac{e^{-x}}{x} = -e^{-x}\left(\frac{1}{x} + \frac{1}{x^2}\right)$

And integration gives

$$\phi(\vec{r}) = kQ\left[\frac{1}{r} - e^{-r/r_0}\left(\frac{1}{r} + \frac{1}{2r_0}\right)\right]$$

as before

3.



To cancel potential of original charge on $x=0$ plane, put an image charge $-q$ at $(-a, b, 0)$.
To cancel potential of original charge on $y=0$ plane, put an image charge $-q$ at $(a, -b, 0)$

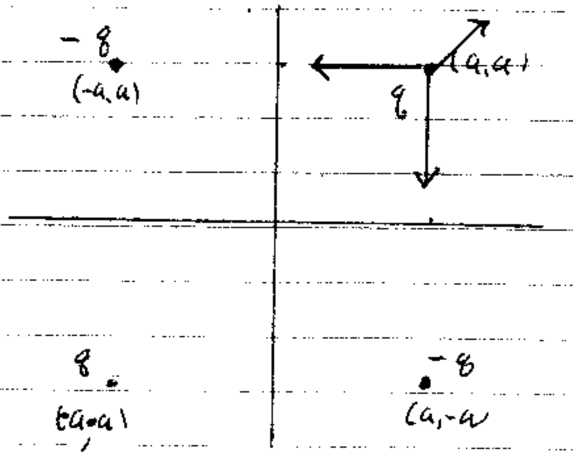
Now we have to cancel potential of image charge $-q$ at $(-a, b, 0)$ on $y=0$, so put another image charge $+q$ at $(-a, -b, 0)$. This also cancels potential of image $-q$ at $(a, -b, 0)$ on $x=0$ plane so we are done

For $\vec{x} = (x, y, z)$

$$
\begin{aligned}
\phi(\vec{x}) = \kappa q \Big\{ & \left[(x-a)^2 + (y-b)^2 + z^2 \right]^{1/2} \\
& - \left[(x+a)^2 + (y-b)^2 + z^2 \right]^{1/2} \\
& - \left[(x-a)^2 + (y+b)^2 + z^2 \right]^{1/2} \\
& + \left[(x+a)^2 + (y+b)^2 + z^2 \right]^{1/2} \Big\}
\end{aligned}
$$

Check $\phi(x=0, y, z) = \phi(x, y=0, z) = 0$

For particle in position $(a, a, 0)$



Net force is in $z=0$ plane and directed towards vertex where planes meet.

$$F = Kq^2 \left\{ \sqrt{2}\,\frac{1}{4a^2} - \frac{1}{8a^2} \right\}$$

$$= \frac{Kq^2}{8a^2}\left(2\sqrt{2} - 1\right)$$