PHY 505

Homework No. 10

1. From lecture: $\mathcal{E}_{HFS} = -\frac{8\pi}{3} |\psi(0)|^2 \frac{\mu_0}{4\pi} \mu_p \mu_e \begin{cases} \frac{1}{4} & S=1 \\ -\frac{3}{4} & S=0 \end{cases}$

For hydrogen ground state: $|\psi(0)|^2 = \frac{1}{\pi a_0^3}$

$$a_0 = 0.529 \times 10^{-11}\, m$$

$$\mu_e = \frac{e\hbar}{m_e} = 2\,(5.79 \times 10^{-11}\ MeV\ T^{-1})$$

$$\mu_p = 2\, g\,(3.15 \times 10^{-14}\ MeV\ T^{-1}), \quad g = 2.79$$

$$\mu_0/4\pi = 10^{-7}\ Newtons/Amp^2$$

$$1\, T = 1\ Newton/Amp\text{-}M$$

$$\therefore \Delta \mathcal{E}_{H.F.S} = \left(\frac{8\pi}{3}\right)(10^{-7})\,(2.79 * 3.15 \times 10^{-14})\,(2 * 5.79 \times 10^{-11})\,\frac{1}{\pi\,(0.529 \times 10^{-10})^3}$$

$$(MeV)^2 \left[\frac{Amp^2 - M^2}{Newtons^2}\right]\left[\frac{Newtons}{Amp^2}\right]\frac{1}{M^3}$$

$$[\Delta\mathcal{E}] = \frac{MeV^2}{Joule}, \qquad 1\ MeV = 1.60 \times 10^{-14}\ Joule$$

$$\Rightarrow \Delta\mathcal{E} = 5.87 \times 10^{-12}\ MeV$$

$$\Delta\mathcal{E} = h\nu = hc/\lambda$$

$$h = (2\pi)(6.58 \times 10^{-22}\ MeV\,s)$$

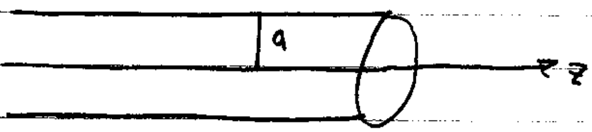
$$c = 3.00 \times 10^{10}\ cm/sec$$

$$\nu = 1.42 \times 10^{9}\ sec^{-1}$$

$$\lambda = 21.1\, cm$$

2. Field of Ideal Solenoid

$$B(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}\, d^3x'$$



$\vec{J} = NI\,\delta(\rho-a)\,\hat{\phi}$

in cylindrical coordinates

Let $\vec{x}$ lie along x-axis in $z=0$ plane

$$\vec{x} = (\rho, 0, 0)$$

$$\vec{x}' = (a\cos\theta,\ a\sin\theta,\ z)$$

Surface current $\vec{K} = NI\, d\vec{l}$

$$d\vec{l} = d\vec{x}' = a\,d\theta\,(-\sin\theta, \cos\theta, 0)$$

$$|\vec{x}-\vec{x}'|^2 = (\rho - a\cos\theta)^2 + a^2\sin^2\theta + z^2$$

$$= z^2+\rho^2+a^2-2\rho a\cos\theta \equiv z^2+b^2$$

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}NI\int_{-\infty}^{\infty}dz\oint\frac{d\vec{l}\times(\vec{x}-\vec{x}')}{|x-x'|^3}$$

$$d\vec{l}\times(\vec{x}-\vec{x}') = z\,(-\cos\phi\,\hat{i} - \sin\phi\,\hat{j}) + (a-\rho\cos\theta)\,\hat{k}$$

Note $\displaystyle\int_{-\infty}^{\infty}\frac{z\,dz}{[z^2+b^2]^{3/2}} = 0$ by antisymmetry in $z\to -z$

$\therefore\ \vec{B}(x) = B(x)\,\hat{k}$

with $$B = \frac{\mu_0 NI}{4\pi}\int_{-\infty}^{\infty}dz\int_0^{2\pi}\frac{a\,d\theta\,(a-\rho\cos\theta)}{[z^2+b^2]^{3/2}}$$

Consider $I = \int_{-\infty}^{\infty} \frac{dz}{[z^2+b^2]^{3/2}}$ Let $z = b \tan\alpha$

$$I = \frac{1}{b^2}\int_{-\pi/2}^{\pi/2} d\alpha \cos\alpha = 2/b^2$$

$$B = \frac{\mu_0 N I a}{2\pi} \int_0^{2\pi} \frac{d\varphi\,(a - \rho\cos\varphi)}{[a^2+\rho^2-2a\rho\cos\varphi]}$$

Use $a^2+\rho^2-2a\rho\cos\varphi = (r_> - r_< e^{i\varphi})(r_> - r_< e^{-i\varphi})$

$$B = \frac{\mu_0 N I a}{2\pi\, r_>^2}\int_0^{2\pi} d\varphi\,\left(a - \rho/2\,(e^{i\varphi} - e^{-i\varphi})\right)\sum_{m_1, m_2=0}^{\infty}\left(\frac{r_<}{r_>}\right)^{m_1+m_2} e^{i(m_1-m_2)\varphi}$$

$$= \frac{\mu_0 N I a}{r_>^2}\left[a\sum_{m=0}\left(\frac{r_<}{r_>}\right)^{2m} - \rho\frac{r_<}{r_>}\sum_{m=0}\left(\frac{r_<}{r_>}\right)^{2m}\right]$$

$$= \mu_0 N I a\left[\frac{a - \rho\, r_</r_>}{r_>^2 - r_<^2}\right]$$

$$= \begin{cases} 0, & \rho > a \quad \text{(outside)} \\ \mu_0 N I, & \rho < a \quad \text{(inside)} \end{cases}$$

3. Recall Dipole Field

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{3(\vec{m}\cdot\hat{x})\hat{x} - \vec{m}}{|\vec{x}|^3}$$

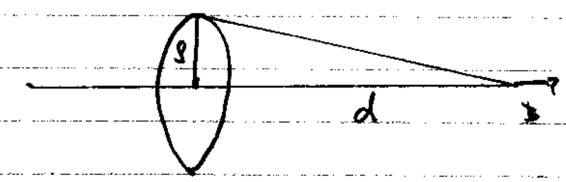
On Axis, $\hat{x} \parallel \vec{m}$ $\quad \vec{B} = \frac{\mu_0}{2\pi} \frac{\vec{m}}{|x|^3}$

For rotating sphere, we have derived in class

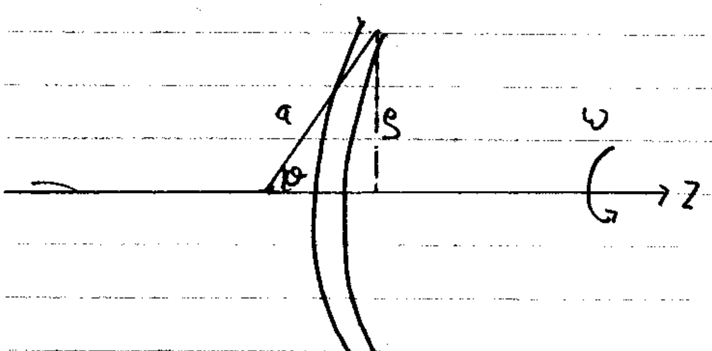
$$\vec{m} = \frac{2 Q a^2 \vec{\omega}}{3}$$

For rotating sphere $Q = 4\pi a^2 \sigma$

For a circular loop carrying current I, Field on Axis is



$$B = \frac{\mu_0 I}{2} \frac{\rho^2}{[\rho^2 + d^2]^{3/2}}$$



Divide sphere into circular strips

$$\Delta I = \frac{\Delta Q}{T} = \frac{\Delta Q\,\omega}{2\pi}$$

$$\Delta Q = \sigma\,(2\pi a \sin\theta)(a\,d\theta')$$

$$= \frac{Q}{2} \sin\theta\, d\theta$$

On Axis $\quad \Delta B_z = \frac{\mu_0}{4\pi} \frac{Q\omega a^2}{2} \underbrace{\int_0^\pi \frac{\sin^3\theta\, d\theta}{[a^2\sin^2\theta + (z - a\cos\theta)^2]^{3/2}}}_{I}$$

$$I = \int_0^{\pi} \frac{(1-\sin^2\theta)\sin\theta\, d\theta}{\{a^2+z^2-2az\cos\theta\}^{3/2}}$$

$$= \int_{-1}^{1} \frac{(1-x^2)\,dx}{\{a^2+z^2-2azx\}^{3/2}}$$

$$= \frac{1}{az}\frac{(1-x^2)}{\{a^2+z^2-2azx\}^{1/2}}\Bigg|_{-1}^{1} + \frac{2}{az}\int_{-1}^{1}\frac{x\,dx}{\{a^2+z^2-2azx\}^{1/2}}$$

$$\overset{\Downarrow}{0}$$

$$= \frac{1}{3(az)^3}\left[\,|z-a|^3 - |z+a|^3 - (a^2+z^2)(|z-a|-|z+a|)\right]$$

$$= \begin{cases} \dfrac{4}{3z^3}, & z > a \\ \dfrac{4}{3a^3} & |z| < a \\ -\dfrac{4}{3z^3} & z < a \end{cases}$$

$$\therefore\ B_z = \frac{\mu_0}{4\pi}\frac{2Q\omega}{3}\begin{cases} \dfrac{1}{a}, & |z|<a \\ \dfrac{a^2}{|z|^3}, & |z|>a \end{cases}$$

For $|z|>a$, we have Dipole field

$$B = \frac{\mu_0}{2\pi}\frac{m}{|z|^3}$$