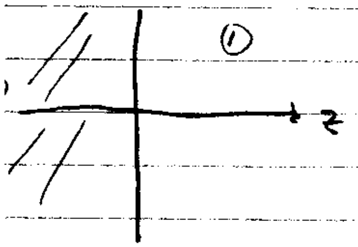
PHY 505

Homework No. 11

Jackson 5.17

It is sufficient to show that the image current solution satisfies the boundary conditions (It obviously satisfies the differential equations

1) $\Delta B_n = 0 \Rightarrow \quad B_z^1(x,y,0) = B_z^2(x,y,0)$



2) $\Delta H_\perp = 0$

$$\left(\frac{1}{\mu_0}\right) B_i^1(x,y,0) = \left(\frac{1}{\mu}\right) B_i^2(x,y,0) \qquad i = x, y$$

1) $$B^1(x,y,0) = \mu_0 \int_{z'>0} \frac{\vec{J}(x') \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \, d^3x' + \mu_0 \int_{z'<0} \frac{\vec{J}^*(x') \; (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} \, d^3x'$$

where $\vec{J}^*$ is the image current given in the text.

$$|x - x'|^2 = (x-x')^2 + (y-y')^2 + z'^2.$$

Evaluating the cross products & collecting terms gives

$$\vec{B}^1(x,y,0) = \frac{\mu_0}{4\pi} \int_{z'>0} \frac{d^3x'}{\left[(x-x')^2 + (y-y')^2 + z'^2\right]^{3/2}} \left\{ \frac{2}{\mu_{R}+1} \left[\left(-z' J_y(\vec{x}') - (y-y') J_z(\vec{x}')\right) \hat{i} + \left(z' J_x(\vec{x}') + (x-x') J_z(\vec{x}')\right) \hat{j} \right\} + \frac{2\mu_R}{\mu_R+1} \left((y-y') J_x(\vec{x}') - (\vec{x} - \vec{x}') J_y(\vec{x}') \right] \right\}$$

2. The ansatz for region 2 ($z<0$) gives

$$\vec{B}^2(x,y,0) = \frac{2\mu_0\mu_R}{\mu_R+1}\int_{z'>0} \frac{d^3x'}{\left[(x-x')^2+(y-y')^2+z'^2\right]^{3/2}}$$

$$\Big\{ \left(-z' J_y(\vec{x}') - (y-y') J_z(\vec{x}')\right)\hat{i}$$

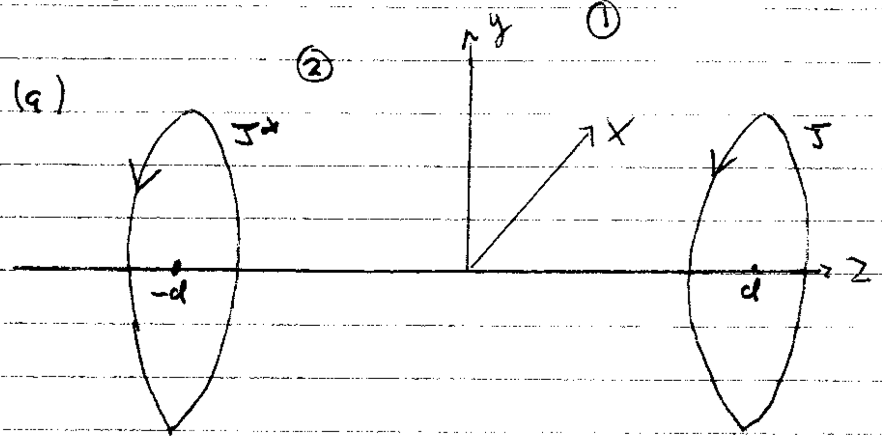
$$+ \left(z' J_x(\vec{x}') + (x-x') J_z(\vec{x}')\right)\hat{j}$$

$$+ \left((x-x') J_y(\vec{x}') - (y-y') J_x(\vec{x}')\right)\hat{k} \Big\}$$

It is straightforward to verify that the boundary conditions are satisfied.

5.18

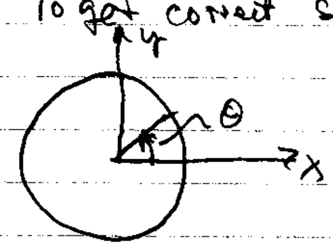
(a)



$$\vec{J}^*(\vec{x}) = \frac{\mu_r - 1}{\mu_r + 1} \vec{J}(x, y, z) \qquad \text{from } 5.17$$

$$\vec{F}_{12} = -\left(\frac{\mu_0}{4\pi}\right) I I^* \oint\oint d\vec{l}_1 \cdot d\vec{l}_2 \frac{\vec{r}}{r^3}, \quad \vec{r} = \vec{x}_1 - \vec{x}_2$$

To get correct sense of circulation, take x-y coordinates as shown



$$\vec{x}_1 = (a\cos\theta_1, a\sin\theta_1, d$$

$$\vec{x}_2 = (a\cos\theta_2, a\sin\theta_2, -d)$$

$$d\vec{l}_1 = a\,d\theta_1\,(-\sin\theta_1, \cos\theta_1, 0)$$
etc for $d\vec{l}_2$

$$d\vec{l}_1 \cdot d\vec{l}_2 = a^2\,d\theta_1\,d\theta_2 \cos(\theta_2 - \theta_1)$$

$$|x_1 - x_2|^2 = 4d^2 + 2a^2(1 - \cos(\theta_2 - \theta_1))$$

$$\vec{F}_{12} = -\frac{\mu_0}{4\pi}\frac{\mu_r - 1}{\mu_r + 1} I^2 a^2 \int_0^{2\pi}\int_0^{2\pi} \frac{d\theta_1\,d\theta_2 \cos(\theta_2 - \theta_1)}{\{4d^2 + 2a^2(1 - \cos(\theta_2 - \theta_1))\}^{3/2}}$$

$$\{a(\cos\theta_1 - \cos\theta_2)\hat{i} + a(\sin\theta_1 - \sin\theta_2)\hat{j} + 2d\hat{k}\}$$

By anti symmetry under $\theta_1 \leftrightarrow \theta_2$ $\quad (F_{12})_x = (F_{12})_y = 0$

ie $\vec{F}_{12} = F\,\hat{k}$

$$F = -\left(\frac{\mu_0}{4\pi}\right) I I^* \, 4\pi a^2 d \int_0^{2\pi} \frac{\cos\theta \, d\theta}{\left[4d^2 + 4a^2(1-\cos^2\theta)\right]^{3/2}}$$

[This cannot be expressed in any simple closed form]

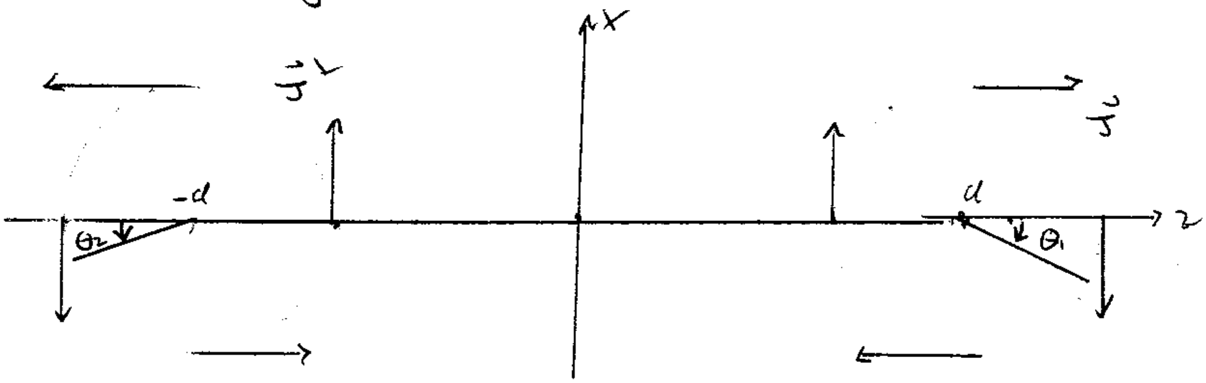
For $d \gg a$ we expand

$$F = -\left(\frac{\mu_0}{4\pi}\right) 2\pi a^2 I I^* \left(\frac{1}{2d}\right)^2 \int_0^{2\pi} d\theta \cos\theta \left[1 + 3\left(\frac{a}{2d}\right)^2(\cos^2\theta - 1) + O\left(\frac{a^4}{d^4}\right) \right]$$

$$= -\left(\frac{\mu_0}{4\pi}\right) (\pi a^2 I)(\pi a^2 I^*) \frac{6}{r^4} \qquad , r = 2d$$

For the current loops $\quad m^{(*)} = \pi a^2 I^{(*)}$

(b) Let the loop lie in the x-z plane. The circulation of $\vec{J}$ & $\vec{J}^*$ is as shown below: The angular coordinates are defined to give the correct sense of circulation



$$J_x^*(x,y,z) = \frac{\mu_r - 1}{\mu_r + 1} J_x(x,y,-z)$$

$$J_z^*(x,y,z) = -\frac{\mu_r - 1}{\mu_r + 1} J_z(x,y,-z)$$

$$\vec{X}_1 = (-a\sin\theta_1,\ 0,\ d + a\cos\theta_1)$$

$$\vec{X}_2 = (-a\sin\theta_2,\ 0,\ -d - a\cos\theta_2)$$

$$d\vec{l}_1 \cdot d\vec{l}_2 = a^2\, d\theta_1\, d\theta_2\, \cos(\theta_1 + \theta_2)$$

$$|\vec{X}_1 - \vec{X}_2|^2 = 4d^2 + 2a^2(1 + \cos(\theta_1+\theta_2)) + 4da(\cos\theta_1 + \cos\theta_2)$$

$$\vec{F} = -\left(\frac{\mu_0}{4\pi}\right) I I^* a^2 \int_0^{2\pi}\int_0^{2\pi} \frac{d\theta_1\, d\theta_2\, \cos(\theta_1+\theta_2)}{[4d^2 + 2a^2(1+\cos(\theta_1+\theta_2)) + 4da(\cos\theta_1 + \cos\theta_2)]^{3/2}}$$

$$\cdot \left\{ a(\sin\theta_2 - \sin\theta_1)\,\hat{i} + (2d + a(\cos\theta_1 + \cos\theta_2))\,\hat{k} \right\}$$

By antisymmetry under $\theta_1 \leftrightarrow \theta_2$, $(F_{12})_x = 0$

$$\vec{F}_{12} = F\,\hat{k}$$

with

$$F = -\left(\frac{\mu_0}{4\pi}\right) I I^* a^2 \iint_0^{2\pi} \frac{d\theta_1\, d\theta_2 \cos(\theta_1+\theta_2)\left[2d + a(\cos\theta_1 + \cos\theta_2)\right]}{\left[4d^2 + 2a^2(1+\cos(\theta_1+\theta_2)) + 4da(\cos\theta_1 + \cos\theta_2)\right]^{3/2}}$$

To expand for $d \gg a$, use

$$[1+x]^{-3/2} = 1 - \tfrac{3}{2}x + \tfrac{1}{2}\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)x^2 + \cdots$$

$$\therefore\ F = -\left(\frac{\mu_0}{4\pi}\right) I I^* \frac{a^2}{(2d)^3} \iint d\theta_1\, d\theta_2 \cos(\theta_1+\theta_2)\left[2d + a(\cos\theta_1 + \cos\theta_2)\right]$$

$$\cdot \left[1 - 3\left(\frac{a}{2d}\right)(\cos\theta_1 + \cos\theta_2) - 3\left(\frac{a}{2d}\right)^2\left[\cos(\theta_1+\theta_2) + 1\right] + \frac{15}{2}\left(\frac{a}{2d}\right)^2(\cos\theta_1 + \cos\theta_2)^2 + \cdots \right]$$

After performing the angular integrals, we get

$$F = -\left(\frac{\mu_0}{4\pi}\right)\underbrace{(\pi a^2 I)}_{m}\underbrace{(\pi a^2 I^*)}_{m^*}\frac{3}{(2d)^4}$$

c) Magnetic Dipole-Dipole Force

$$F_{12} = \nabla\left(\vec{m}_1 \cdot \vec{B}_2(x_1)\right)$$

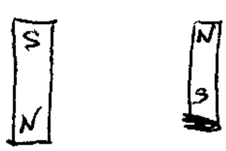
$$\vec{B}_2(\vec{x}_1) = \left(\frac{\mu_0}{4\pi}\right)\left[\frac{3(\vec{m}_2\cdot\hat{r})\,\hat{r} - \vec{m}_2}{r^3}\right] \qquad \vec{r} = \vec{x}_1 - \vec{x}_2$$

Part a) Corresponds to $\vec{m}_1 \parallel \vec{m}_2$ and $\parallel$ to $\vec{r}$

$$\vec{m}_1 \cdot \vec{B}_2 = \left(\frac{\mu_0}{4\pi}\right)\frac{2\,m_1 m_2}{r^3}$$

$$F_{12} = \nabla(\vec{m}_1\cdot\vec{B}_2) = -\left(\frac{\mu_0}{4\pi}\right)\frac{6\,m_1 m_2}{r^4}\,\hat{r} \qquad \text{attraction}$$

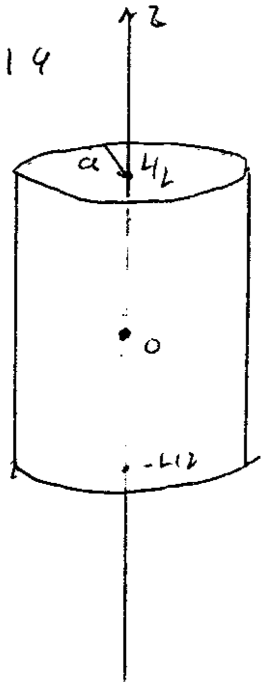
Part b) corresponds to $\vec{m}_1$ Anti $\parallel$ to $\vec{m}_2$ & both $\perp$ $\vec{r}$



$$\vec{m}_1 \cdot \vec{B}_2 = \left(\frac{\mu_0}{4\pi}\right)\frac{m_1 m_2}{r^3}$$

$$F_{12} = -\left(\frac{\mu_0}{4\pi}\right)\frac{3\,m_1 m_2}{r^4}\cdot\hat{r} \qquad \text{attraction}$$

3.5.14



Inside cylinder $\vec{M} = M_0 \hat{z}$ = constant

No free currents: $\nabla \times \vec{H} = 0$
$H = -\nabla \phi_m$

$$\vec{B} = \mu_0 [\vec{H} + \vec{M}]$$

$$\nabla \cdot \vec{B} = 0 \Rightarrow \nabla^2 \phi_M = \nabla \cdot M$$

$$\phi_M(\vec{x}) = \frac{-1}{4\pi} \int \frac{d^3x' \, \nabla \cdot M(x')}{|\vec{x} - \vec{x}'|}$$

Only contribution comes from ends of cylinder
where $X' = (r\cos\theta, r\sin\theta, \pm L/2)$

$$\vec{x} = (0, 0, z)$$

$$\phi_m = \frac{M_0}{4\pi} \int_0^{2\pi} d\theta \int_0^a r\,dr \left[\frac{1}{[r^2 + (z - L/2)^2]^{1/2}} - \frac{1}{[r^2 + (z + L/2)^2]^{1/2}} \right]$$

$$= \frac{M_0}{2} \left\{ [r^2 + (z - L/2)^2]^{1/2} - [r^2 + (z + L/2)^2]^{1/2} \right\} \Big|_{r=0}^{r=a}$$

$$= \frac{M_0}{2} \left\{ [a^2 + (z - L/2)^2]^{1/2} - [a^2 + (z + L/2)^2]^{1/2} + |z + L/2| - |z - L/2| \right\}$$

$$\phi_m = \frac{M_0}{2} \left\{ [a^2 + (z - L/2)^2]^{1/2} - [a^2 + (z + L/2)^2]^{1/2} + \begin{Bmatrix} L \\ 2z \\ -L \end{Bmatrix} \right\} \quad \text{for} \quad \begin{matrix} z > L/2 \\ |z| < L/2 \\ z < -L/2 \end{matrix}$$

On axis $\vec{H} = H\hat{z}$, $\quad H = -\frac{\partial}{\partial z}\Phi_m$

$$\therefore H = \frac{M_0}{2}\left\{ \frac{z + L/2}{[(z+L/2)^2 + a^2]^{1/2}} - \frac{z - L/2}{[(z-L/2)^2+a^2]^{1/2}} \right\}$$

$$- M\,\Theta(L/2 - |z|)$$

$$B = \mu_0(H + M) = \frac{\mu_0 M_0}{2}\left\{ \quad '' \quad \right\}$$

c) $\quad \frac{H}{M_0} = \frac{B}{\mu_0 M_0} - M\,\Theta(L/2 - |z|)$

Note: for $z > L/2$, this is field of dipole with magnitude $p = (\pi a^2) M_0 L$

$\therefore \lim_{z \to \infty} B = \left(\frac{\mu_0}{4\pi}\right)\frac{2p}{z^3}$ as required