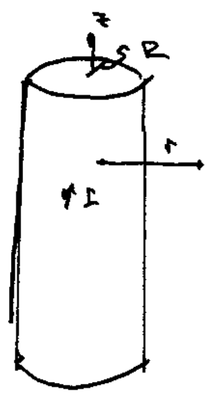
PHY 505
Homework # 12

$$W = \frac{1}{2}\int \vec{J}\cdot\vec{A}\, dV = \frac{1}{2} L I^2$$ for closed circuit with total current I



For infinitely long wire with uniform current density in z-direction, we find by Ampere's Law.

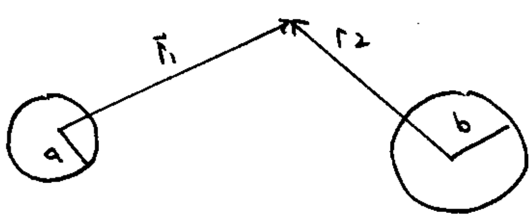
$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi} \qquad r > a$$

$$\vec{B} = \frac{\mu_0 I r}{2\pi a^2}\hat{\phi} \qquad r < a$$

From $\vec{B} = \nabla\times\vec{A}$, we can take $B_\phi = -\frac{\partial A_z}{\partial r}$ here

$$\Rightarrow \quad A_z = \begin{cases} -\frac{\mu_0 I}{2\pi}\{\ln(r/R) + c\} & r > a \\ -\frac{\mu_0 I}{4\pi}\, r^2/R^2 & , r < a \end{cases}$$

Continuity at $r = R \Rightarrow \quad C = \frac{1}{2}$



For 2 wires with current up in $R = a$ & down in $R = b$, we have

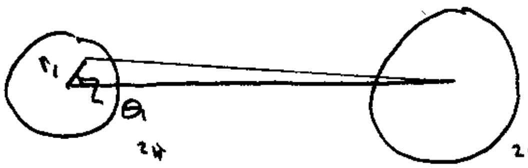
$$A_z = -\frac{\mu_0 I}{4\pi}\Big[\{\ln(\frac{r_1^2}{a^2}) + 1\}\,\Theta(r_1 - a)$$
$$- \{\ln(\frac{r_2^2}{b^2}) + 1\}\,\Theta(r_2 - a)$$
$$+ (\frac{r_1}{a})^2\,\Theta(a - r_1)$$
$$- (r_2/b)^2\,\Theta(b - r_2)\Big]$$

Note: we can add overall constant to A_z without changing W.

Then $$W = \frac{\mu_0 I^2}{8\pi} \int dz \left\{ \frac{1}{\pi a^2} \int_0^{2\pi} d\theta_1 \int_0^{a} r_1 dr_1 \left[2 \ln\left(\frac{r_2}{b}\right) + 1 - \left(\frac{r_1}{a}\right)^2 \right] \right.$$

$$\left. + \frac{1}{\pi b^2} \int_0^{2\pi} d\theta_2 \int_0^{b} r_2 dr_2 \left[2 \ln\left(\frac{r_1}{b}\right) + 1 - \left(\frac{r_2}{b}\right)^2 \right] \right\}$$

$\int dz$ = Length of cylinder. Drop this integral to get Energy per unit length $\mathcal{W} = \frac{1}{2} \mathcal{L} I^2$



$$\int_0^{2\pi} d\theta_1 \ln r_2^2 = \int_0^{2\pi} d\theta_1 \ln\left(d^2 + r_1^2 - 2 d r_1 \cos\theta_1\right)$$

{ See Class notes} $$= \int_0^{2\pi} d\theta_1 \left\{ \ln d^2 - 2 \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r_1}{d}\right)^n \cos(n\theta) \right\}$$

$$= 2\pi \ln d^2$$

i.e. $$\mathcal{L} = \frac{\mu_0}{4\pi} \left\{ 2 \ln\left(\frac{d}{b}\right) + 1 - \frac{1}{2} \right.$$

$$\left. + 2 \ln\left(\frac{d}{a}\right) + 1 - \frac{1}{2} \right\}$$

$$= \frac{\mu_0}{4\pi} \left\{ 2 \ln\left(\frac{d^2}{ab}\right) + 1 \right\}$$