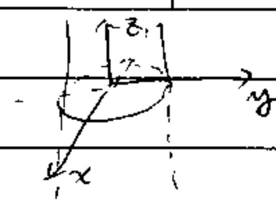
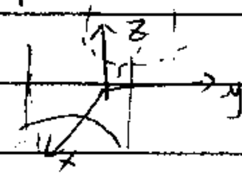
HM #1 Solution

1. a). $u = \text{constant} \Rightarrow \dfrac{x^2}{a^2\cosh^2 u} + \dfrac{y^2}{a^2\sinh^2 u} = 1$ → ellipse in x-y plane.



$v = \text{constant} \Rightarrow \dfrac{x^2}{a^2\cos^2 v} - \dfrac{y^2}{a^2\sin^2 v} = 1$

↳ hyperbola in x-y plane.



b). $ds^2 = (dx)^2 + (dy)^2 + (dz)^2$

$$= \left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right)^2 + \left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)^2 + dz^2$$

$$= a^2(\sinh^2 u\cos^2 v + \cosh^2 u\sin^2 v)[(du)^2 + (dv)^2] + dz^2$$

$$= a^2(\sinh^2 u + \sin^2 v)[(du)^2 + (dv)^2] + dz^2$$

(use $\dfrac{\partial x}{\partial u} = a\sinh u\cos v$, $\dfrac{\partial x}{\partial v}$ . . . etc.)

c). By b). $h_u = h_v = h = a\sqrt{\sinh^2 u + \sin^2 v}$, $h_z = 1$

$$\nabla\cdot\vec{V} = \frac{1}{h_u h_v h_z}\left[\frac{\partial}{\partial u}(h_v h_z V_u) + \frac{\partial}{\partial v}(h_u h_z V_v) + \frac{\partial}{\partial z}(h_u h_v V_z)\right]$$

$$= \frac{1}{h^2}\left[\frac{\partial}{\partial u}(hV_u) + \frac{\partial}{\partial v}(hV_v)\right] + \frac{\partial}{\partial z}V_z$$

$$\nabla^2\phi = \frac{1}{h_u h_v h_z}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_z}{h_u}\frac{\partial\phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_z}{h_v}\frac{\partial\phi}{\partial v}\right) + \frac{\partial}{\partial z}\left(\frac{h_u h_v}{h_z}\frac{\partial\phi}{\partial z}\right)\right]$$

$$= \frac{1}{h^2}\left[\frac{\partial^2}{\partial u^2}\phi + \frac{\partial^2}{\partial v^2}\phi\right] + \frac{\partial^2}{\partial z^2}\phi$$

$$\nabla\times\vec{V} = \frac{1}{h_v h_z}\left[\frac{\partial}{\partial v}(h_z V_z) - \frac{\partial}{\partial z}(h_v V_v)\right]\hat{u} + \frac{1}{h_u h_z}\left[\frac{\partial}{\partial z}(h_u V_u) - \frac{\partial}{\partial u}(h_z V_z)\right]\hat{v}$$

$$+ \frac{1}{h_u h_v}\left[\frac{\partial}{\partial u}(h_v V_v) - \frac{\partial}{\partial v}(h_u V_u)\right]\hat{z}$$

$$= \left(\frac{1}{h}\frac{\partial}{\partial u}V_z - \frac{\partial}{\partial z}V_v\right)\hat{u} + \left(\frac{\partial}{\partial z}V_u - \frac{1}{h}\frac{\partial}{\partial u}V_z\right)\hat{v}$$

$$+ \frac{1}{h^2}\left[\frac{\partial}{\partial u}(hV_v) - \frac{\partial}{\partial v}(hV_u)\right]\hat{z}$$

2. a). Let $\vec{V} = \vec{R}\phi$, $\vec{R}$ constant vector.

$$\oint d\vec{l}\cdot\vec{V} = \int ds\,\vec{n}\cdot[\nabla\times(\vec{R}\phi)] \qquad \text{use } \nabla\times(\vec{R}\phi) = \nabla\phi\times\vec{R} + \overset{0}{\cancel{\phi\nabla\times\vec{R}}}$$

$$= \int ds\,\vec{n}\cdot[\nabla\phi\times\vec{R}] \qquad , A\cdot(B\times C) = C\cdot(A\times B)$$

$$= \int ds\,\vec{R}\cdot[\vec{n}\times\nabla\phi]$$

$$\oint d\vec{l}\cdot\vec{V} = \int d\vec{l}\cdot(\vec{R}\phi)$$

$$= \vec{R}\cdot\int d\vec{l}\,\phi$$

Since $\vec{R}$ is arbitrary, $\int d\vec{l}\,\phi = \int ds\,(\vec{n}\times\nabla\phi)$

b). Let $\vec{K} = \vec{R}\times\vec{V}$

$$\oint d\vec{l}\cdot\vec{K} = \int ds\,\vec{n}\cdot[\nabla\times(\vec{R}\times\vec{V})]$$

$$= \int ds\,(\vec{n}\times\nabla)\cdot(\vec{R}\times\vec{V}) \qquad \text{use } A\cdot(B\times C) = (A\times B)\cdot C$$

$$= \int ds\,\vec{R}\cdot[\vec{V}\times(\vec{n}\times\nabla)] \qquad \text{use } \nearrow$$

$$= -\vec{R}\cdot\int ds\,[(\vec{n}\times\nabla)\times\vec{V}] \qquad ①$$

$$\oint d\vec{l}\cdot\vec{K} = \int d\vec{l}\cdot(\vec{R}\times\vec{V})$$

$$= \vec{R}\cdot\int(\vec{V}\times d\vec{l})$$

$$= -\vec{R}\cdot\int d\vec{l}\times\vec{V} \qquad ②$$

Since $\vec{R}$ is arbitrary, $\int d\vec{l}\times\vec{V} = \int ds\,(\vec{n}\times\nabla)\times\vec{V}$.

c). Let the $\vec{V}$ in b) be $\nabla\phi$ $\Rightarrow$

$$\int d\vec{l}\times\nabla\phi = \int ds\,[(\vec{n}\times\nabla)\times\vec{\nabla}]\phi$$

$$= -\int ds\,[\nabla\times(\vec{n}\times\nabla)]\phi \qquad \text{use } A\times(B\times C) = (A\cdot C)B - (A\cdot B)C.$$

$$= -\int ds\,[\vec{n}\nabla^2 - (\vec{n}\cdot\nabla)\nabla]\phi$$

$$= \int ds\,[(\vec{n}\cdot\nabla)\nabla - \vec{n}\nabla^2]\phi$$

3. $\oint \vec{E}\cdot ds = \dfrac{Q}{\varepsilon_0}$,

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \to$ integrate this by part

$$Q(r) = \int 4\pi\rho_0(r')\,r'^2 dr' = \begin{cases} 4\pi\rho_0\int_0^r e^{-\frac{r'}{b}}\,r'^2 dr' & r \le b \\ 4\pi\rho_0\int_0^b e^{-\frac{r'}{b}}\,r'^2 dr' & r \ge b \end{cases}$$

By symmetric property, we know $\vec{E} = E\hat{r}$

$$E = \frac{Q}{4\pi r^2\varepsilon_0} = \frac{1}{\varepsilon_0 r^2}\times\begin{cases} \rho_0 b^3\left[2 - e^{-\frac{r}{b}}\left(\frac{r^2}{b^2} + 2\frac{r}{b} + 2\right)\right] & r \le b \\ \rho_0 b^3\left(2 - \frac{5}{e}\right) & r > b. \end{cases}$$