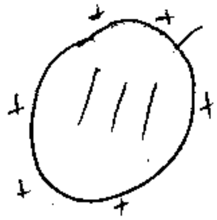
HW #2. Solution:

1. a).



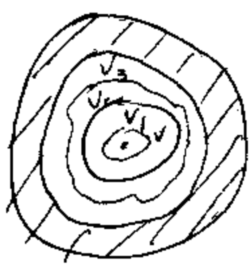
Since the Conductor must reach its equilibrium, So $\vec{E}_{inside} = 0$

$$\Rightarrow \nabla \cdot \vec{E}_{inside} = \frac{\rho}{\varepsilon_0} = 0$$

$$\Rightarrow \rho_{inside} = 0.$$

So Charge Lie entirely on its surface!

b).

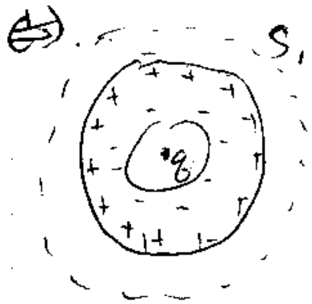


q

If there is $\vec{E}$ field inside the Cavity, one can draw equipotential surfaces like the figure shown. Choose one of the surfaces S_1 as Gaussians surface. Considering $\vec{E}$ field is perpendicular to S_1 everywhere, and can't flip its directions along S_1 (either point out / point in everywhere) $\Rightarrow \int_{S_1} \vec{E} \cdot ds \neq 0$.

this means $\rho = \varepsilon_0 \int_{S_1} \vec{E} \cdot ds \neq 0$!

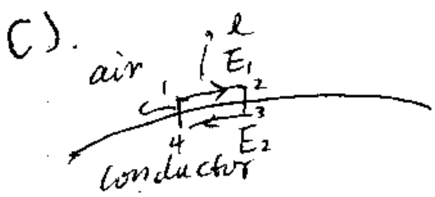
It violate our assumption that $\rho = 0$ everywhere inside the cavity. So $\vec{E}$ field can't exists inside the Cavity b).



If q is in the Cavity. We choose S_1 in the left graph as Gaussian surface.

$$\int_{S_1} \vec{E} \cdot ds = \frac{q}{\varepsilon_0} \neq 0 \;\Rightarrow\; \vec{E} \text{ can't be } 0 \text{ everywhere!}$$

C).



$$\oint_l E \cdot d\vec{l} = 0 \Rightarrow \int_{1\to 2} \vec{E}_1 d\vec{l} + \int_{2\to 3} \vec{E}_1 \cdot d\vec{l} + \int_{3\to 4} \vec{E}_2 \cdot d\vec{l} + \int_{4\to 1} E_1 \cdot d\vec{l} = 0$$

Choose the distance from 3 to 2 and 1 to 4 be infinitesimal, ~~by consi.~~

and consider $E_2 = 0$ (inside conductor)

$$\Rightarrow \int_{1\to 2} \vec{E}_1 \cdot d\vec{l} = 0$$

we always can chose the loop to satisfy this.

If $l_{1\to 2}$ is small enough, we can treat $\vec{E}_1$ as a constant in this region.

$$\Rightarrow \vec{E}_1 \cdot \vec{l}_{1\to 2} = 0$$

$$\Rightarrow \vec{E}_1 = 0. \quad \Rightarrow \text{No parallel component!}$$

1b) Alternative Proof

Green Identity!

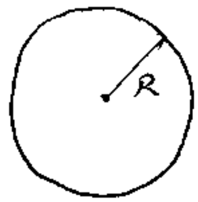
$$\int_V \Phi \nabla^2 \Phi \, d^3x + \int |\nabla \Phi|^2 d^3x = \oint \Phi \frac{\partial \Phi}{\partial n} ds$$

Let S be the inside surface of the conductor where $\Phi = \Phi_S =$ constant

Then $\oint \Phi \frac{\partial \Phi}{\partial n} ds = \Phi_S \oint \frac{\partial \Phi}{\partial n} ds = 0$ (no charge inside)

Also $\nabla^2 \Phi = 0 \Rightarrow \nabla \Phi = 0$ inside.

1.3. a).



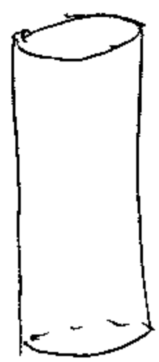
Let. $\rho(r,\theta,\varphi) = A\,\delta(r-R)$

since. $\iint \rho\, r^2 \sin\theta\, dr\, d\theta\, d\varphi = Q$

$$\Rightarrow A = \frac{Q}{4\pi R^2}$$

$$\Rightarrow \rho = \frac{Q}{4\pi R^2}\delta(r-R)$$

b).



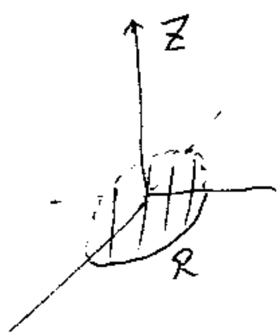
Let $\rho(r,\theta,z) = A\,\delta(r-b)$

since $\dfrac{\int \rho\, r\, dr\, d\theta\, dz}{z} = \lambda$

$$\Rightarrow A = \frac{\lambda}{2\pi b}$$

$$\Rightarrow \rho = \frac{\lambda}{2\pi b}\delta(r-b)$$

c)



Let $\rho(r,\theta,z) = A\,\delta(z)\,\theta(R-r)$

$$\theta(R-r) = \begin{cases} 1 & R \geq r \\ 0 & R < r \end{cases}$$

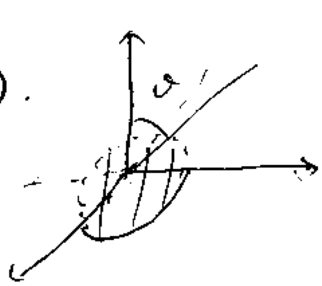
de

since $\int \rho\, r\, d\theta\, dr\, dz = Q$

$$\Rightarrow A = \frac{Q}{\pi R^2}$$

$$\Rightarrow \rho = \frac{Q}{\pi R^2}\delta(z)\,\theta(R-r)$$

D).



Jackson (1.2)

Let $\rho(r,\theta,\varphi) = A\,\dfrac{\delta(\theta-\frac{\pi}{2})}{r}\,\theta(R-r)$

$\Rightarrow \cancel{A=} \int \rho\, r^2 \sin\theta\, dr\, d\theta\, d\varphi = Q$

$\Rightarrow A = \dfrac{Q}{\pi R^2} \quad \Rightarrow \rho = \dfrac{Q}{\pi R^2}\,\dfrac{\delta(\theta-\frac{\pi}{2})}{r}\,\theta(R-r)$

1.5. $\rho = \varepsilon_0 [-\nabla^2 \phi]$

$$= -\frac{q}{4\pi} \nabla^2 \left[\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right)\right]$$

$$= -\frac{q}{4\pi} \nabla^2 \left[\frac{e^{-\alpha r}}{r} + \frac{\alpha}{2} e^{-\alpha r}\right]$$

since $\nabla^2 (AB) = B\nabla^2 A + A\nabla^2 B + 2(\nabla A)\cdot(\nabla B)$

$$\rho = -\frac{q}{4\pi}\left[\nabla^2\left(\frac{\alpha}{2}e^{-\alpha r}\right) + \frac{1}{r}\nabla^2 e^{-\alpha r} + e^{-\alpha r}\nabla^2 \frac{1}{r} + 2\times\left(\nabla \frac{1}{r}\right)\cdot\left(\nabla e^{-\alpha r}\right)\right]$$

$$= -\frac{q}{4\pi}\left[-4\pi e^{-\alpha r}\delta(r) + \left(\frac{\alpha}{2}+\frac{1}{r}\right)\nabla^2 e^{-\alpha r} + \frac{2\alpha}{r^2} e^{-\alpha r}\right]$$

$$\nabla^2 e^{-\alpha r} = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} e^{-\alpha r} = \frac{1}{r}\frac{\partial^2}{\partial r^2} r$$

$$= \frac{-2\alpha}{r} e^{-\alpha r} + \alpha^2 e^{-\alpha r}$$

$$\Rightarrow \rho = -\frac{q}{4\pi}\left[\frac{\alpha^3}{2} e^{-\alpha r} - 4\pi e^{-\alpha r}\delta(r)\right]$$

$$= q\,\delta(r)\, e^{-\alpha r} - \frac{q\alpha^3}{8\pi} e^{-\alpha r}.$$

$r \to 0$ the first part is much more important!

$$\int_{V\to 0} q\,\delta(r)\, e^{-\alpha r}\, dV = q$$

So it shows that a positive charge sitting in origion, proton!

$r > 0$. the second part is more important.

$$\int_{V\to\infty} -\frac{q}{8\pi}\alpha^3 e^{-\alpha r}\, dV = -q$$

~~it~~ It corresponds to the electron cloud.