HW #3

1. $\int_{r_0-\varepsilon}^{r_0+\varepsilon} r^2 dr (L.H.S) = \int_{r_0-\varepsilon}^{r_0+\varepsilon} r^2 dr \left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{\hat{L}^2}{r^2} \right) \frac{1}{|\vec{x}-\vec{x}_0|}$

$\{ \hat{L}^2 = - \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \right] \}$

$= r^2 \frac{\partial}{\partial r} \frac{1}{|\vec{x}-\vec{x}_0|} \Big|_{r_0-\varepsilon}^{r_0+\varepsilon} - \int_{r_0-\varepsilon}^{r_0+\varepsilon} \hat{L}^2 \frac{1}{|\vec{x}-\vec{x}_0|} dr$

$\{ \frac{1}{|\vec{x}-\vec{x}_0|} = 4\pi \sum_\ell \sum_m \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\theta_0,\phi_0) Y_{\ell m}(\theta,\phi) \}$

$= 4\pi \sum_\ell \sum_m \frac{1}{2\ell+1} Y_{\ell m}^*(\theta_0,\phi_0) Y_{\ell m}(\theta,\phi) \left[-(\ell+1) \frac{(r_0+\varepsilon)^2 r_0^\ell}{(r_0+\varepsilon)^{\ell+2}} \right.$

$\left. - \ell \frac{(r_0-\varepsilon)^{\ell-1}(r_0-\varepsilon)^2}{r_0^{\ell+1}} \right] - \sum_\ell \sum_m 4\pi \frac{1}{2\ell+1} \times \ell(\ell+1) Y_{\ell m}^* Y_{\ell m}$

$\times \int_{r_0-\varepsilon}^{r_0+\varepsilon} \frac{r_<^\ell}{r_>^{\ell+1}} dr$

$\hookrightarrow$ since $\frac{r_<^\ell}{r_>^{\ell+1}}$ is continuous ~~everywhere~~ around the region $[r_0-\varepsilon, r_0+\varepsilon]$, this integral must be zero as $\varepsilon \to 0$.

$\varepsilon \to 0$ $\quad = -4\pi \sum_\ell \sum_m Y_{\ell m}^*(\theta_0,\phi_0) Y_{\ell m}(\theta,\phi) \qquad ①$

$\int_{r_0-\varepsilon}^{r_0+\varepsilon} r^2 dr (R.H.S) = \int_{r_0-\varepsilon}^{r_0+\varepsilon} -4\pi \frac{\delta(r-r_0)\delta(\theta-\theta_0)\delta(\varphi-\varphi_0)}{r^2\sin\theta} r^2 dr$

$= -4\pi \frac{\delta(\theta-\theta_0)\delta(\varphi-\varphi_0)}{\sin\theta_0} \qquad ②$

Since ① = ②, one get.

$\sum_\ell \sum_m Y_{\ell m}^*(\theta_0,\phi_0) Y_{\ell m}(\theta,\phi) = \frac{\delta(\theta-\theta_0)\delta(\phi-\phi_0)}{\sin\theta_0}$

$f(\theta,\phi) = \int f(\theta',\phi')\, \delta(\theta-\theta')\delta(\phi-\phi')\, d\theta' d\phi'$

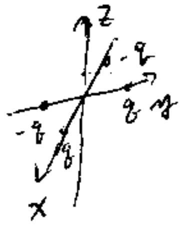
$= \int f(\theta',\phi') \frac{\delta(\theta-\theta')\delta(\phi-\phi')}{\sin\theta'} \sin\theta'\, d\theta' d\phi'$

$= \sum_{\ell,m} \int Y_{\ell m}^*(\theta',\phi') Y_{\ell m}(\theta,\phi) f(\theta',\phi') \sin\theta'\, d\theta' d\phi'$

$= \sum_{l,m} f_{l,m} Y_{lm}(\theta, \phi)$

$$f_{l,m} = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \, f(\theta,\phi) Y_{lm}^*(\theta,\phi).$$

3. a).



Charge : $q = \sum_i q_i = 0$

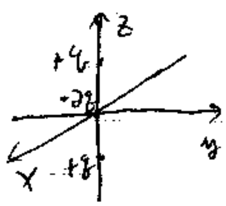
dipole : $P = \sum_i \vec{x}_i q_i = qa\hat{x} + (-q)a(-\hat{x}) + qa\hat{y} + (-q)a(-\hat{y}) = 2aq(\hat{x} + \hat{y})$

quadrople : $Q_{ij} = \sum_{ij} (3x_i x_j - r^2 \delta_{ij}) q$

$Q_{i \neq j} = 0$ since all charges are located in axis (X or Y axis)

$Q_{33} = Q_{11} = Q_{22} = \cancel{\sum_{ij}} (3x_i x_i - a^2)(q - q) = 0$, $i = 1,2,3$

~~$Q_{33} = 0$ since no charge located out side x-y ax plane~~ .



Charge : $q = 0$.

dipole : $P = qa(\hat{z} - \hat{z}) - 2q \times 0 = 0$

quadrople : $Q_{i \neq j} = 0$. since all charges are located in $\hat{z}$ axis.

$Q_{11} = -a^2(q+q) = -2a^2 q = Q_{22}$

$Q_{33} = (3a^2 - a^2)q + (3a^2 - a^2)q = 4a^2 q$

b). By (4.10) in Jackson, one get the potential generated by a quadrople is

$$\phi(x) = \frac{1}{4\pi\varepsilon_0} \times \frac{1}{2} \sum_{ij} Q_{ij} \frac{x_i x_j}{r^5}$$

$$= \frac{1}{4\pi\varepsilon_0} \times \frac{1}{2} \left(-2a^2 q x^2 - 2a^2 q y^2 + 4a^2 q z^2\right) \frac{1}{r^5}$$

$$= \frac{qa^2}{4\pi\varepsilon_0 r^3} (3\cos^2\theta - 1)$$

In ~~z~~ x-y plane $\theta = \frac{\pi}{2}$.

$$\phi(x) = \frac{-qa^2}{4\pi\varepsilon_0 r^3}$$

(C). By superposition principle:

$$\phi = \frac{2q}{4\pi\varepsilon_0\sqrt{r^2+a^2}} - \frac{2q}{4\pi\varepsilon_0 r}$$

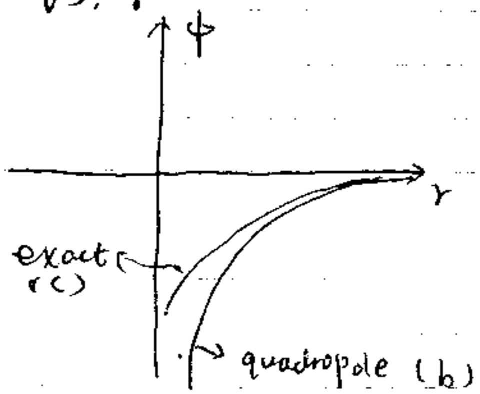
$$= \frac{-q}{2\pi\varepsilon_0 r}\left(1 - \frac{1}{\sqrt{1+\left(\frac{a}{r}\right)^2}}\right)$$

for $r \gg a$, one get a first order expansion

$$\phi(r) = \frac{-q}{2\pi\varepsilon_0 r}\left[1 - \cancel{\frac{1}{2}}\, 1 + \frac{1}{2}\left(\frac{a}{r}\right)^2\right]$$

$$= \frac{-q a^2}{4\pi\varepsilon_0 r^3} \quad \longrightarrow \text{same as (b)}$$

plot ϕ vs. r



4. a) $\vec{F} = \int \rho(x') \vec{E}(x') d^3x'$ ①

Expand: $\vec{E}(\vec{x}) = \vec{E}(0) + x_i \left[\frac{\partial}{\partial x_i} \vec{E}(\vec{x})\right]\Big|_{x=0} + \frac{1}{2} x_i x_j \left[\frac{\partial^2}{\partial x_i \partial x_j} \vec{E}(\vec{x})\right]$

Consider the contribution of each of these terms to the force

i) $\int \rho(x') \vec{E}(0) d^3x' = q \vec{E}(0)$

ii) $\int \rho(x') x'_i \frac{\partial}{\partial x_i} \vec{E}(x)\big|_{x=0} d^3x' = p_i \frac{\partial}{\partial x_i} \vec{E}(x)\big|_{x=0}$

Now $(\vec{p} \cdot \nabla)(\vec{E}) = \hat{e}_j p_i \frac{\partial}{\partial x_i} E_j$

But $\nabla \times \vec{E} = 0 \Rightarrow \frac{\partial}{\partial x_i} E_j = \frac{\partial}{\partial x_j} E_i$

& $(\vec{p} \cdot \nabla)\vec{E}\big|_{x=0} = \hat{e}_j p_i \frac{\partial E_i}{\partial x_j}\Big|_{x=0} = \nabla(\vec{p} \cdot \vec{E})\big|_{x=0}$

iii) $\frac{1}{2} \int \rho(x') x'_i x'_j \left(\frac{\partial^2}{\partial x_i \partial x_j} E\right)\Big|_{x=0} d^3x$

$= \frac{1}{6} Q_{ij} \partial_i \partial_j \vec{E}\big|_{x=0} + \frac{1}{6} \int \rho(x) x^2 \sum_i \frac{\partial^2 \vec{E}}{\partial x_i^2}\Big|_{x=0}$

(1) $\sum_i \frac{\partial^2}{\partial x_i^2} \vec{E} = \sum_{ik} \hat{e}_k \frac{\partial^2}{\partial x_i \partial x_k} E_i = \nabla(\nabla \cdot \vec{E}) = 0$

(2) $Q_{ij} \frac{\partial^2 \vec{E}}{\partial x_i \partial x_j} = \sum_{ijk} \hat{e}_k Q_{ij} \frac{\partial^2}{\partial x_i \partial x_k} E_j$

$= \nabla\left(\sum_{ij} Q_{ij} \frac{\partial E_j}{\partial x_i}\right)$

4 b) From Jackson (4.13) Fields at $\vec{x}_1$ due to dipole $\vec{p}_2$ at $\vec{x}_2$ is

$$E(\vec{x}_1) = K \frac{3(\vec{x}_1 - \vec{x}_2)(\vec{p}_2 \cdot (\vec{x}_1 - x_2)) - |\vec{x}_1 - \vec{x}_2|^2 \vec{p}_2}{|\vec{x}_1 - x_2|^5}$$

$K = \frac{1}{4\pi\epsilon_0}$

Force on dipole $\vec{p}_1$ at $\vec{x}_1$ due to $\vec{p}_2$ is from part a

$$\vec{F}_{12} = \nabla_1 (\vec{p}_1 \cdot E(x_1))$$

$$= K \nabla \left\{ \frac{3\{p_1 \cdot (\vec{x}_1 - \vec{x}_2)\}\{\vec{p}_2 \cdot (\vec{x}_1 - \vec{x}_2)\}}{|\vec{x}_1 - \vec{x}_2|^5} - \frac{\vec{p}_1 \cdot \vec{p}_2}{|\vec{x}_1 - \vec{x}_2|^3} \right\}$$

$$= K \quad \frac{3\{\vec{p}_1 \; p_2 \cdot (\vec{x}_1 - \vec{x}_2) + \vec{p}_2 \; \vec{p}_1 \cdot (\vec{x}_1 - \vec{x}_2)\}}{|\vec{x}_1 - x_2|^5}$$

$$- 15 \frac{(p_1 \cdot (\vec{x}_1 - \vec{x}_2))(\vec{p}_2 \cdot (x_1 - x_2))(\vec{x}_1 - \vec{x}_2)}{|\vec{x}_1 - x_2|^7}$$

$$+ \frac{3(\vec{p}_1 \cdot \vec{p}_2)(\vec{x}_1 - \vec{x}_2)}{|\vec{x}_1 - \vec{x}_2|^5}$$

$$= \frac{K}{|\vec{x}_1 - \vec{x}_2|^4} \{ 3\{\vec{p}_1 (\vec{p}_2 \cdot \vec{n}) + \vec{p}_2 (\vec{p}_1 \cdot \hat{n}) - 15 (\vec{p}_1 \cdot n)(\vec{p}_2 \cdot n) \hat{n}$$

$$+ 3 (\vec{p}_1 \cdot \vec{p}_2) \hat{n} \}$$

To find $\vec{F}_{21}$ Let $\begin{pmatrix} \hat{n} \to -\hat{n} \\ \vec{x}_1 \leftrightarrow \vec{x}_2 \end{pmatrix} \Rightarrow \vec{F}_{21} = -\vec{F}_{12}$

5. (a) By Jackson (4.1), ($r \gg r'$ → condition of expansion)

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \sum_l \sum_m \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta\ \phi)}{r^{l+1}} \qquad ①$$

$$q_{lm} = \int Y^*_{lm}(\theta', \phi') r'^l \rho(\vec{x}') d^3x',$$

Since $\sin^2\theta = \frac{4\sqrt{\pi}}{3}(Y_{00} - \sqrt{\frac{1}{5}} Y_{20})$ and

$$\int Y^*_{lm}(\theta', \phi') Y_{l'm'}(\theta', \phi') \sin\theta' d\theta' d\phi' = \delta_{ll'} \delta_{mm'}$$

In ①, only two terms survive, for $l = 0, 2$ and $m = 0$.

$$q_{00} = \int_0^\infty \frac{1}{\sqrt{4\pi}} \frac{1}{64\pi} r^2 e^{-r} \sin^2\theta\, d^3x$$

$$= \sqrt{\frac{1}{4\pi}}$$

$$q_{20} = -\int Y^*_{20}(\theta', \phi') r'^2 \frac{1}{64\pi} r'^2 e^{-r'} \frac{4\sqrt{\pi}}{3} \frac{1}{\sqrt{5}} Y_{20} \sin\theta\, r'^2 dr' d\theta' d\phi'$$

$$= \frac{1}{64\pi} \times \frac{4}{3} \times \sqrt{\frac{\pi}{5}} \times \int_0^\infty r^6 e^{-r} dr$$

$$= -15\sqrt{\frac{1}{5\pi}}$$

$$\text{so, } \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left(\sqrt{\frac{1}{4\pi}} \times 4\pi \frac{Y_{00}}{r} - 15\sqrt{\frac{1}{5\pi}} \times \frac{4\pi}{5} \frac{Y_{20}(\theta, \phi)}{r^3}\right)$$

By (3.57) in Jackson,

$$Y_{00} = \sqrt{\frac{1}{4\pi}} P_0, \qquad Y_{20} = \sqrt{\frac{5}{4\pi}} P_2$$

Finally, one get

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{P_0}{r} - \frac{6}{r^3} P_{20}(\theta, \phi)\right]$$

for correct dimension, one can write it as

$$\Phi(\vec{x}) = \frac{\rho_0 r_0^3}{4\pi\varepsilon_0}\left[\frac{P_0}{r} - 6\frac{r_0^2}{r^3} P_{20}(\theta\ \phi)\right]$$

(b). For $r > r'$ one can use the result of (a) to decode the potential.

For $r \ll r'$ (around origin)

$$\phi(x) = \frac{1}{\varepsilon_0} \sum_{\ell m} \frac{1}{2\ell+1} \underbrace{\left[\int Y_{\ell m}^*(\theta',\phi') \frac{1}{r'^{\ell+1}} \rho(\vec{x}')\, d^3x'\right]}_{q'_{\ell m}} r^\ell Y_{\ell m}(\theta,\phi)$$

By the same reason as (a) only q_{00}, q_{20} survive.

$$q'_{00} = \int Y_{00}^* \frac{1}{64\pi} r^2 e^{-r} \frac{1}{r} \frac{4\sqrt{\pi}}{3} Y_{00}\, r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= \frac{1}{64\pi} \times \frac{4\sqrt{\pi}}{3} \times \int_0^\infty r^3 e^{-r}\, dr$$

$$= 6 \times \frac{1}{64\pi} \times \frac{4\sqrt{\pi}}{3} = \frac{1}{8\sqrt{\pi}}$$

$$q'_{20} = -\int Y_{20}^*(\theta',\phi') \frac{1}{r^3} \frac{1}{64\pi} r^2 e^{-r} \frac{4\sqrt{\pi}}{3}\sqrt{\frac{1}{5}}\, Y_{20}(\theta',\phi')\, r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= -\int_0^\infty r e^{-r}\, dr \times \frac{1}{64\pi} \times \frac{4\sqrt{\pi}}{3}\sqrt{\frac{1}{5}}$$

$$= -\frac{1}{48\sqrt{5\pi}}$$

$$\Rightarrow \phi(x) = \frac{1}{4\pi\varepsilon_0}\left[4\pi \times \frac{1}{8\sqrt{\pi}} \frac{1}{\sqrt{4\pi}} - \frac{1}{48} \times \sqrt{\frac{1}{5\pi}} \times 4\pi \times \frac{1}{5} \times r^2 \sqrt{\frac{5}{4\pi}} P_{20}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{4} - \frac{r^2}{120} P_{20}\right)$$

for correct dimension

$$\phi(x) = \frac{\rho_0 r_0^3}{4\pi\varepsilon_0}\left(\frac{1}{4r_0} - \frac{r^2}{120 r_0^3} P_{20}\right)$$

General case.

$$\phi(x) = \int_0^r \frac{\rho(\vec{r}')}{4\pi\varepsilon_0 |r-r'|} d^3x' + \int_r^\infty \frac{\rho(\vec{r}')}{4\pi\varepsilon_0 |r-r'|} d^3x'$$

$$= \frac{1}{\varepsilon_0}\left(q_{00}(r)\frac{Y_{00}}{r} + \frac{1}{5} q_{20}(r) \frac{Y_{20}}{r^3} + q'_{00}(r) Y_{00} + \frac{1}{5} q'_{20}(r)\, r^2 Y_{20}(\theta,\phi)\right)$$

$$q_{00}(r) = \frac{4\sqrt{\pi}}{3}\int_0^r r^4 e^{-r} dr \frac{1}{64\pi}$$

$$= \frac{4}{3}\sqrt{\pi}\left(-r^4 e^{-r} - 4r^3 e^{-r} - 12 r^2 e^{-r} - 24 r e^{-r} - 24 e^{-r} + 24\right)\frac{1}{64\pi}$$

$$q_{20}(r) = \frac{1}{48\pi}\sqrt{\frac{\pi}{5}}\int_0^r r^6 e^{-r} dr$$

$$= \frac{1}{48\pi}\sqrt{\frac{\pi}{5}}\left(-r^6 e^{-r} - 6r^5 e^{-r} - 30 r^4 e^{-r} - 120 r^3 e^{-r} - 360 r^2 e^{-r} \ldots\right.$$

$$q'_{00}(r) = \frac{4\sqrt{\pi}}{3} \times \frac{1}{64\pi}\int_r^\infty r^3 e^{-r} dr$$

$$q'_{20}(r) = \ldots \int^\infty r' e^{-r'} dr$$