HW # 4

1. a). $S = F(h,0) + F(-h,0) + F(0,h) + F(0,-h)$

$$F(\pm h,0) = F(0,0) + F_x(0,0)\times(\pm h) + \frac{1}{2!}F_{xx}(0,0)(\pm h)^2 + \cdots$$

Clearly $F(h,0) + F(-h,0)$ will kill odd-h-order terms (i.e. $h, h^3, h^5 \ldots$

It's also true for $F(0,h) + F(0,-h)$.

So. $$S = 4F(0,0) + \frac{2}{2}F_{xx}(0,0)h^2 + \frac{2}{2}F_{yy}(0,0)h^2 + \frac{2}{4!}(F_{xxxx} + F_{yyyy})h^4 + O(h^6)$$

$$= 4F(0,0) + \nabla^2 F(0,0)h^2 + \frac{h^4}{12}(F_{xxxx} + F_{yyyy}) + O(h^6)$$

b). $S = F(-h\ h) + F(h\ -h) + F(h\ h) + F(-h\ -h)$.

$$F(h,h) = F(0,0) + h(\partial_x + \partial_y)F(0,0) + \frac{h^2}{2}(\partial_x + \partial_y)^2 F(0,0) + \cdots$$
$$F(-h-h) = F(0,0) - h(\partial_x + \partial_y)F(0,0) + \frac{h^2}{2}(\partial_x + \partial_y)^2 F(0,0) - \cdots$$

Clearly, all odd order terms die. similiarly for.

$$F(h,-h) = F(0,0) + h(\partial_x - \partial_y)F(0,0) + \frac{h^2}{2}(\partial_x - \partial_y)^2 F(0,0) + \cdots$$
$$F(-h,h) = F(0,0) - h(\partial_x - \partial_y)F(0,0) + \frac{h^2}{2}(\partial_x - \partial_y)^2 F(0,0) - \cdots$$

all odd order terms die. by add them up.

So. $$S = 4F(0,0) + h^2[(\partial_x + \partial_y)^2 + (\partial_x - \partial_y)^2]F(0,0) + \frac{1}{12}h^4[(\partial_x + \partial_y)^4 + (\partial_x - \partial_y)^4]F(0,0) + O(h^6)$$

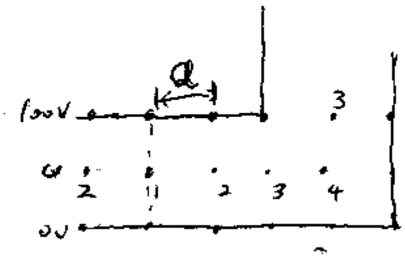
$$= 4F(0,0) + 2h^2\nabla^2 F + \frac{1}{12}h^4[2\partial_x^4 + 2\partial_y^4 + 2\times 6(\partial_x^2\partial_y^2)]F(0,0)$$

$$= 4F(0,0) + 2h^2\nabla^2 F + \frac{1}{6}h^4[3(\partial_x^2 + \partial_y^2)^2 - 2(\partial_x^4 + \partial_y^4)]F(0,0)$$

$$= 4F(0,0) + 2h^2\nabla^2 F + \frac{1}{2}h^4(\partial_x^2 + \partial_y^2)^2 F(0,0) - \frac{1}{3}h^4(\partial_x^4 + \partial_y^4)F(0,0)$$

$$= 4F(0,0) + 2h^2\nabla^2 F + \frac{1}{2}h^4\nabla^2\nabla^2 F - \frac{1}{3}h^4(F_{xxxx} + F_{yyyy})$$

2. a). For Cross case



$$\phi_1 = 25 + \tfrac{1}{2}\phi_2$$

$$\phi_2 = 25 + \tfrac{1}{4}(\phi_1 + \phi_3)$$

$$\phi_3 = 25 + \tfrac{1}{4}(\phi_2 + \phi_4)$$

$$\phi_4 = \tfrac{1}{2}\phi_3$$

Solve these combined equations directly one get.

$\phi_1 = 48.889$
$\phi_2 = 47.778$
$\phi_3 = 42.222$
$\phi_4 = 21.111$

one can also use iterative method to get these value. (See the maple Command text).

Improved case:

$$\phi_1 = \tfrac{2}{5}\phi_2 + 30$$
$$\phi_2 = \tfrac{1}{5}(\phi_1 + \phi_3) + 30$$
$$\phi_3 = \tfrac{1}{5}(\phi_2 + \phi_4) + 25 + \tfrac{1}{20}\phi_3$$
$$\phi_4 = \tfrac{2}{5}\phi_3 + 5.$$

Solve these equations directly, one get

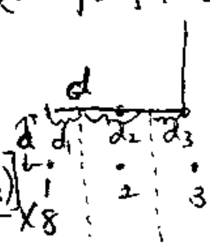
$\phi_1 = 49.2109$
$\phi_2 = 48.0273$
$\phi_3 = 40.9257$
$\phi_4 = 21.3703$

One can also use iterative method. (see - Maple Command text).

b). $C = \dfrac{Q}{V}$, $Q = \sum_i A_i \dfrac{E_i}{} \times \varepsilon_0$, $A_i = d_i \ell$

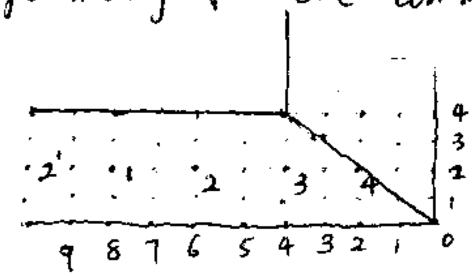
$\Rightarrow \dfrac{C}{\ell} = \sum_i E_i d_i \varepsilon_0$ (d_i is the space range for a given E_i value)

$$E_i = \frac{100 - \phi_i}{a},$$



$$\Rightarrow \frac{C}{\ell} = \frac{\left[\tfrac{1}{2}\times(100-\phi_1) + (100-\phi_2) + \tfrac{1}{2}(100-\phi_3)\right]}{100} \times 8$$
$$= 1.07 \times 8 = 8.56\ \varepsilon_0\ F/m$$

c. see maple command text.
the geometry for the command is



Boundary condition: $\phi[i,j] = \phi[j,i]$ for $i,j \le 4$

$\phi[i,7] = \phi[i,9]$ $\quad i = 1,2,3,4$

$\phi[i,j] = \frac{1}{4}(\phi[i+1,j] + \phi[i,j+1] + \phi[i-1,j] + \phi[i,j-1])$

for $i = 1,2,3$, $j = 1,2,3,\cdots 8$.

see the result.

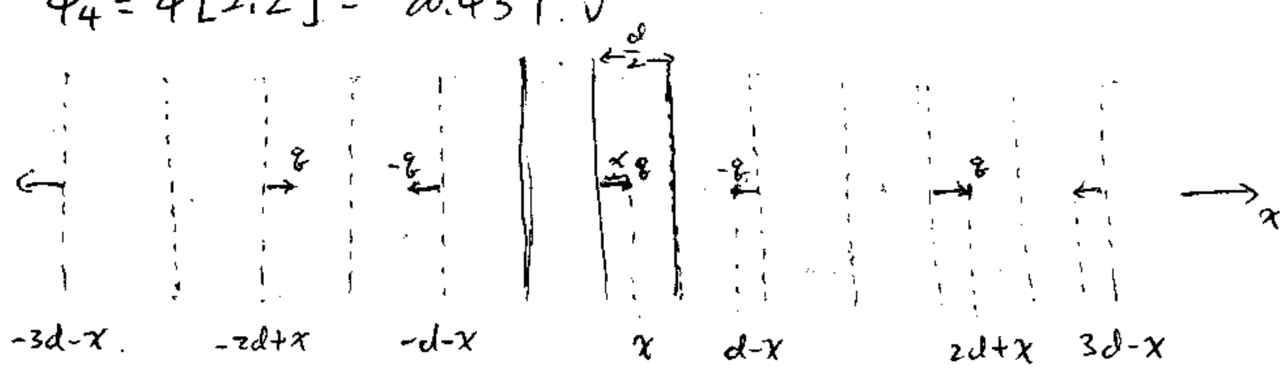
$\phi_1 = \phi[2,8] = 49.396$ ✓

$\phi_2 = \phi[2,6] = 47.589$ ✓

$\phi_3 = \phi[2,4] = 39.874$ ✓

$\phi_4 = \phi[2,2] = 20.437$ ✓

3.



So. $F(x) = \sum_{n\neq 0} \frac{-q^2 k n \times 2d}{|n \times 2d|^3} \hat{x} + \sum_{n=-\infty}^{\infty} \frac{kq^2[(2n+1)d - 2x]}{|(2n+1)d - 2x|^3} \hat{x}$

clearly, ~~for~~ the first term is zero since $\sum_{n=1}^{\infty}$ and $\sum_{n=-\infty}^{n=-1}$ cancel each other

So.

$$F(x) = \sum_{n=-\infty}^{\infty} \frac{k q^2 [(2n+1)d - 2x]}{|[2n+1]d - 2x|^3} \hat{x}$$

$$= \sum_{n=0}^{\infty} \frac{q^2 k}{[(2n+1)d - 2x]^2} + \sum_{n=-1}^{-\infty} \frac{-q^2 k}{[(2n+1)d - 2x]^2}$$

$$= \sum_{n=0}^{\infty} k q^2 \left[\frac{1}{[(2n+1)d - 2x]^2} - \frac{1}{[(2n+1)d + 2x]^2} \right]$$

$$= \sum_{n=0}^{\infty} k q^2 \frac{8x(2n+1)d}{[(2n+1)^2 d^2 - 4x^2]^2}$$

(b). $x \sim 0$, Expand around $x = 0$.

$$F = \left[\sum_{n=0}^{\infty} 8 k q^2 \frac{1}{[(2n+1)d]^3} \right] x .$$

As a result $x \to 0$, $F \to 0$. and F is proportional to x around origin.

$x \sim +\frac{d}{2}$. Let $x = +\frac{d}{2} - x'$ $\quad x' \sim 0$

$$F = \sum_n k \frac{q^2[(2n+1)d - 2x]}{|(2n+1)d - 2x|^3}$$

$$= \sum_n k \frac{q^2[(2n\cancel{+1})d + 2x']}{|(2nd) + 2x'|^3}$$

$$= \sum_{n=1}^{\infty} k q^2 \frac{1}{|2nd + 2x'|^2} + \sum_{-\infty}^{-1} k q^2 \frac{-1}{|2nd + 2x'|^2} + \frac{kq^2}{4x'^2}$$

$$= \sum_{n=1}^{\infty} k q^2 \left[\left(\frac{1}{2nd}\right)^2 - 2\frac{2x'}{(2nd)^3} \right] - \sum_{n=-\infty}^{-1} k q^2 \left[\frac{1}{(2nd)^2} - 2\frac{2x'}{(2nd)^3} \right] + \frac{kq^2}{4x'^2}$$

$$= k\frac{q^2}{4x'^2} - 4kq^2 \sum_{n=1}^{\infty} \frac{x'}{(2nd)^3} + \sum_{n=-\infty}^{-1} 4kq^2 \frac{x'}{(2nd)^3}$$

$$= k\frac{q^2}{4x'^2} - 8kq^2 \sum_{n=1}^{\infty} \frac{x'}{(2nd)^3}$$

As $x' \to 0$, the force goes to infinity. and the parts beyond $n=0$ have very tiny effects on $F(x)$.

c) Unless the charged droplet is very close to one plate, on the center line, the image charges exert a force $\sim \frac{Kq^2}{d^2}$ ($K = 9.0\times10^9\ Nm^2/C^2$)

In Millikan's expt.+, the voltage between the plates was $\sim 5000V$ and the separation was $d = 16\ mm$

The electric force was $F_{PLATE} = \frac{V}{d} q$

$\therefore$ the ratio of forces was $\frac{F_{IMAGE}}{F_{PLATE}} \sim \frac{Kq}{Vd}$

With $q = ne$, $e = 1.60\times10^{-19}C$
n is a small integer

the ratio is $\sim n\times10^{-11}$ which is negligible!

+ http://dbhs.wvusd.k12.ca.us/AtomicStructure.html

```
> phi1:=50;phi2:=50;phi3:=50;phi4:=25;
```

$$\phi 1 := 50$$

$$\phi 2 := 50$$

$$\phi 3 := 50$$

$$\phi 4 := 25$$

```
> for i from 1 to 100 do
  phi1:=25+phi2/2;phi2:=25+(phi1+phi3)/4;phi3:=25+(phi2+phi4)/4;phi4
  :=phi3/2;end do;
> evalf(phi1);
>
```

$$48.88888889$$

```
> evalf(phi2);
```

$$47.77777778$$

```
> evalf(phi3);
```

$$42.22222222$$

```
> evalf(phi4);
```

$$21.11111111$$

```
>
```

```
> phi1:=50;phi2:=50;phi3:=50;phi4:=25;
> for i from 1 to 100 do
  phi1:=1/5*(2*phi2+100)+10;phi2:=1/5*(phi1+phi3+100)+10;phi3:=1/5*(
  100+phi2+phi4)+5+phi3/20;phi4:=2/5*phi3+5;end do;
> evalf(phi1);
> evalf(phi1);
```

$$49.21094161$$

$$49.21094161$$

```
> evalf(phi2);
```

$$48.02735402$$

```
> evalf(phi3);
```

$$40.92582851$$

```
> evalf(phi4);
```

$$21.37033140$$

```
>
```

```
> i:=0;j:=0;phi1:=array(0..4,0..9);phi:=array(0..4,0..9);for i from
  0 to 4 do for j from i to 9 do phi[i,j]:=i*25;end do;end do;
```

$$i := 0$$

$$j := 0$$

$$\phi 1 := \mathrm{array}(0 .. 4, 0 .. 9, [\,])$$

$$\phi := \mathrm{array}(0 .. 4, 0 .. 9, [\,])$$

```
> for i from 0 to 4 do for j from 0 to 4 do phi[j,i]:=phi[i,j];end
  do;end do;for i from 0 to 4 do for j from 0 to 9 do
  phi1[i,j]:=phi[i,j]; end do;end do;print(phi);for k from 1 to 100
  do for i from 0 to 4 do for j from 0 to 9 do
  phi[i,j]:=phi1[i,j];end do;end do; for i from 1 to 3 do for j from
  i to 8 do
  phi1[i,j]:=(phi[i+1,j]+phi[i-1,j]+phi[i,j+1]+phi[i,j-1])/4;end
  do;end do;for l from 1 to 4 do for m from 1 to 4 do
  phi1[m,l]:=phi1[l,m];end do;end do;for n from 1 to 3 do
  phi[n,9]:=phi[n,7];end do;end do;

> evalf(phi[2,8]);
```

$$49.06518970$$

```
> evalf(phi[2,6]);
```

$$47.51452365$$

```
> evalf(phi[2,4]);
```

$$39.85550449$$

```
> evalf(phi[2,2]);
>
```

$$20.42864032$$

```
>
```