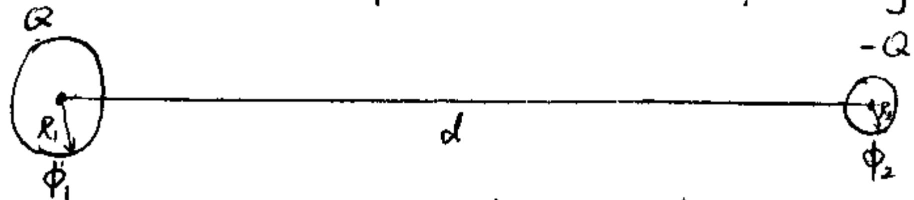
HW # 6

1\. By definition, the capacitance of a system of two conductor carrying equal and opposite charges is the ratio of the charge on one conductor to the potential difference between them. In our case, the setup is shown as following



$$\phi_1 = K\frac{Q}{R_1} - K\frac{Q}{d-R_1} - K\frac{Q'}{R_1} + K\frac{Q'}{R_1 - \frac{R_1^2}{d}} \quad , \quad Q' = \frac{R_1}{d}Q$$

$$= K\frac{Q}{R_1} - K\frac{Q}{d} \qquad ①$$

$$\phi_2 = K\frac{-Q}{R_2} - K\frac{-Q}{d}$$
$$= K\frac{Q}{d} - K\frac{Q}{R_2} \qquad ②$$

From ①, ②, one get

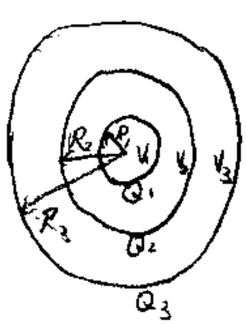
$$C = \frac{Q}{\phi_1 - \phi_2} = \frac{1}{K}\left[\frac{R_1+R_2}{R_1R_2} - \frac{2}{d}\right]^{-1}$$

$$= \frac{\tilde{C}_0}{K}\left[1 - \frac{2\tilde{C}_0}{d}\right]^{-1}$$

$$= 4\pi\varepsilon_0\tilde{C}_0\left(1 + \frac{2\tilde{C}_0}{d}\right) + O\left(\frac{\tilde{C}_0}{d}\right) \quad , \quad \tilde{C}_0 = \frac{R_1R_2}{R_1+R_2}$$

As $d \gg R_1 > \tilde{C}_0$, one get
$$C \doteq 4\pi\varepsilon_0\tilde{C}_0\left(1 + \frac{2\tilde{C}_0}{d}\right)$$

2\.



From the configuration shown in the sketch, one get

$$V_3 = K(Q_1 + Q_2 + Q_3)/R_3 \qquad ①$$
$$V_2 = V_3 + \int_{R_2}^{R_3}\left(\frac{Q_1+Q_2}{R^2}\right)K\,dR \qquad ②$$
$$V_1 = V_2 + \int_{R_1}^{R_2}\frac{Q_1}{R}K\,dR \qquad ③$$

by using Gaussian law

From equations ①~③, one can extract that

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = K \begin{pmatrix} \frac{1}{R_1}, & \frac{1}{R_2}, & \frac{1}{R_3} \\ \frac{1}{R_2}, & \frac{1}{R_2}, & \frac{1}{R_3} \\ \frac{1}{R_3}, & \frac{1}{R_3}, & \frac{1}{R_3} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} \qquad ④$$

the 3×3 matrix is C^{-1} as defined in Jackson 1.11.
Inverse ④, one get·

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \frac{1}{K} \begin{pmatrix} \frac{R_1R_2}{R_2-R_1}, & \frac{R_1R_2}{R_1-R_2}, & 0 \\ \frac{R_1R_2}{R_1-R_2}, & \frac{(R_3-R_1)R_2^2}{(R_2-R_3)(R_1-R_2)}, & \frac{R_3R_2}{R_2-R_3} \\ 0, & \frac{R_2R_3}{R_2-R_3}, & \frac{-R_3^2}{R_3-R_2} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \qquad ⑤$$

Consequently, one get the coefficients of capacitance from the 3× matrix C,

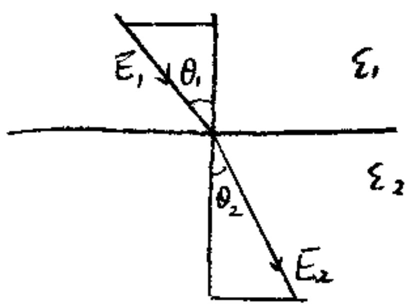
$$C_{11} = \frac{R_1R_2}{R_2-R_1} = -C_{12} = -C_{21}, \quad C_{13} = C_{31} = 0, \quad C_{23} = C_{32} = \frac{R_2R_3}{R_2-R_3}$$

$$C_{33} = R_3^2/(R_3-R_2), \quad C_{22} = (R_3-R_1)R_2^2/(R_2-R_3)(R_1-R_2)$$

$$W = \frac{1}{2}\sum_{ij} Q_i C^{-1}_{ij} Q_j = \frac{1}{2}(-Q, 2Q, -Q) \begin{pmatrix} \frac{1}{R_1}, & \frac{1}{R_2}, & \frac{1}{R_3} \\ \frac{1}{R_2}, & \frac{1}{R_2}, & \frac{1}{R_3} \\ \frac{1}{R_3}, & \frac{1}{R_3}, & \frac{1}{R_3} \end{pmatrix} \begin{pmatrix} -Q \\ 2Q \\ -Q \end{pmatrix} K$$

$$= \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_3}\right)$$

3.



By boundary condition of Maxwell equation, one know

$$\left.\begin{aligned} E_1 \sin\theta_1 &= E_2 \sin\theta_2 \\ \varepsilon_1 E_1 \cos\theta_1 &= \varepsilon_2 E_2 \cos\theta_2 \end{aligned}\right\}$$

$$\Rightarrow \tan\theta_2 = \frac{\varepsilon_2}{\varepsilon_1}\tan\theta_1$$

$$\Rightarrow \theta_2 = \arctan\left[\frac{\varepsilon_2}{\varepsilon_1}\tan\theta_1\right]$$

$$E_2 = \frac{E_1 \sin\theta_1}{\sin\theta_2}$$

$$= E_1 \sin\theta_1 / \sin\left[\tan^{-1}\left(\frac{\varepsilon_2}{\varepsilon_1}\tan\theta_1\right)\right]$$

$$= E_1 \sqrt{\sin^2\theta_1 + \left(\frac{\varepsilon_1}{\varepsilon_2}\cos\theta_1\right)^2}$$