PHY505
Homework No. 7 Solutions

1. Jackson 4.8

   There are 3 regions

   (a) $r > b$, ϵ_0
   (b) $a < r < b$, ϵ
   (c) $r < a$, ϵ_0

   Let the axis of the cylinders be the z-axis. Take the applied field to point in the x-direction. That is, $\vec{E}_0 = E_0\hat{x}$ corresponding to a potential

$$\phi_0 = -E_0 x = -E_0 r\cos\theta\,.$$

   The potential in each of the regions can be expanded as

$$\begin{aligned}
\phi_1 &= -E_0 r\cos\theta + A_0^{(1)} + B_0^{(1)}\ln r \\
&\quad \sum_{n=1}^{\infty}\{B_n^{(1)}r^{-n}\cos n\theta + D_n^{(1)}r^{-n}\sin n\theta\} \\
\phi_2 &= A_0^{(2)} + B_0^{(2)}\ln r + \\
&\quad \sum_{n=1}^{\infty}\{(A_n^{(2)}r^n + B_n^{(2)}r^{-n})\cos n\theta + \\
&\quad (C_n^{(2)}r^n + D_n^{(2)}r^{-n})\sin n\theta\} \\
\phi_3 &= A_0^{(3)} + \sum_{n=1}^{\infty}\{A_n^{(3)}r^n\cos n\theta + C_n^{(3)}r^n\sin n\theta\}
\end{aligned}$$

   There is no $\ln r$ term in ϕ_3 because it would give a $1/r$ singularity in the electric field at the origin. Since the potential is defined only up to an overall constant, we can set $A_0^{(1)} = 0$.

The boundary conditions are:

$$\begin{aligned}
\phi_1(b,\theta) &= \phi_2(b,\theta) \\
\phi_2(a,\theta) &= \phi_3(a,\theta) \\
\epsilon_0 \frac{\partial \phi_1}{\partial r}\bigg|_{r=b} &= \epsilon \frac{\partial \phi_2}{\partial r}\bigg|_{r=b} \\
\epsilon \frac{\partial \phi_2}{\partial r}\bigg|_{r=a} &= \epsilon_0 \frac{\partial \phi_3}{\partial r}\bigg|_{r=a}
\end{aligned}$$

Since $\sin n\theta$ and $\cos n\theta$ are orthogonal on $0 < \theta < 2\pi$, these equations must be satisfied separately by the coefficients of each trigonometric term (including $1 = \cos 0\theta$). Except for the $\cos\theta$ terms the boundary conditions give sets of 4 independent *homogeneous* equations in 4 unknowns. Only the trivial solutions with all coefficients equal to 0 are allowed.

For the $\cos\theta$ terms we have inhomogeneous equations. Relabeling the coefficients

$$\begin{aligned}
B_1^{(1)} &= B_1 \\
A_1^{(2)} &= A_2 \\
B_1^{(2)} &= B_2 \\
A_1^{(3)} &= A_3
\end{aligned}$$

the boundary conditions can be written as

$$\begin{aligned}
B_1 - A_2 b^2 - B_2 &= E_0 b^2 \\
B_1 + \kappa A_2 b^2 - \kappa B_2 &= -E_0 b^2 \\
A_2 a^2 + B_2 - A_3 a^2 &= 0 \\
\kappa A_2 a^2 - \kappa B_2 - A_3 a^2 &= 0
\end{aligned}$$

where $\kappa = \epsilon/\epsilon_0$. The solution is

$$\begin{aligned}
B_1 &= \frac{2(\kappa^2-1)(b^2-a^2)}{\Delta} E_0 b^2 \\
A_2 &= \frac{-2(\kappa+1)}{\Delta} E_0 b^2
\end{aligned}$$

$$
\begin{aligned}
B_2 &= \frac{-2(\kappa-1)a^2}{\Delta}E_0 b^2 \\
A_3 &= \frac{-4\kappa}{\Delta}E_0 b^2
\end{aligned}
$$

where $\Delta = (\kappa+1)^2 b^2 - (\kappa-1)^2 a^2$.

We have two checks on these results.

a) If $\kappa = 1$, there is no interface, and we obtain $B_1 = B_2 = 0$, and $A_2 = A_3 = -E_0$. Thus, $\phi = -E_0 x$ everywhere.

b) If $b = a$, $B_1 = 0$, and $A_3 = -E_0$.

In general we have

$$
\begin{aligned}
\phi_1 &= -E_0 x \left\{ 1 - \left[\frac{2(\kappa^2-1)(b^2-a^2)b^2}{\Delta}\right]\left[\frac{1}{x^2+y^2}\right]\right\} \\
\phi_2 &= -E_0 x \left\{ \left[\frac{2(\kappa+1)b^2}{\Delta}\right] + \left[\frac{2(\kappa-1)a^2b^2}{\Delta}\right]\left[\frac{1}{x^2+y^2}\right]\right\} \\
\phi_3 &= -E_0 x \left[\frac{4\kappa b^2}{\Delta}\right]
\end{aligned}
$$

Defining

$$
\vec{V} = \nabla\left[\frac{x}{x^2+y^2}\right] = \frac{(y^2-x^2)\hat{x} - 2xy\hat{y}}{(x^2+y^2)^2},
$$

the electric fields in the three regions are given by

$$
\begin{aligned}
\vec{E}_1 &= \vec{E}_0 - \left[\frac{2(\kappa^2-1)(b^2-a^2)b^2}{\Delta}\right] E_0 \vec{V} \\
\vec{E}_2 &= \left[\frac{2(\kappa+1)b^2}{\Delta}\right]\vec{E}_0 + \left[\frac{2(\kappa-1)a^2b^2}{\Delta}\right] E_0 \vec{V} \\
\vec{E}_3 &= \left[\frac{4\kappa b^2}{\Delta}\right]\vec{E}_0
\end{aligned}
$$

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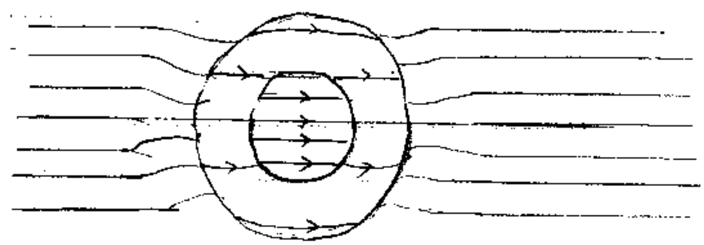
4.8 b. For $b = 2a$

$$\Delta = 4(k+1)^2 a^2 - (k-1)^2 a^2 = (3k^2 + 10k + 3) a^2$$

$$\vec{E}_1 = \vec{E}_0 - \left[\frac{24(k^2-1)a^4}{\Delta}\right] E_0 \vec{V}$$

$$\vec{E}_2 = \left[\frac{8(k+1)a^2}{\Delta}\right] \vec{E}_0 + \left[\frac{8(k-1)a^4}{\Delta}\right] E_0 \vec{V}$$

$$\vec{E}_3 = \left[\frac{16ka^2}{\Delta}\right] \vec{E}_0$$



c. $a \to 0$

$$\Delta = (k+1)^2 b^2$$

$$\vec{E}_1 = \vec{E}_0 - \left[\frac{2(k-1)b^2}{(k+1)}\right] E_0 \vec{V}$$

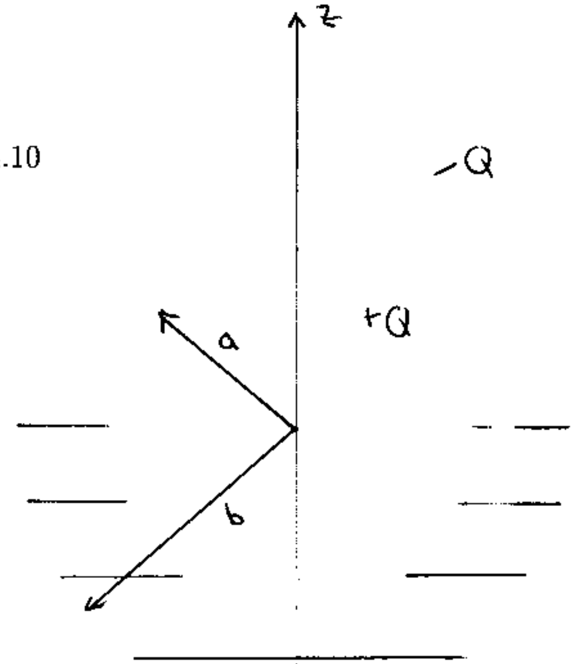
$$\vec{E}_2 = \left[\frac{2}{(k+1)}\right] \vec{E}_0$$

$b \to \infty$

$$\vec{E}_2 = \left[\frac{2}{(k+1)}\right] \vec{E}_0 + \left[\frac{2(k-1)a^2}{(k+1)^2}\right] E_0 \vec{V}$$

$$\vec{E}_3 = \left[\frac{4k}{(k+1)^2}\right] \vec{E}_0$$

2. Jackson 4.10



(a) & (b) Choose the z-axis to be perpendicular to the plane where the 2 hemispheres meet as shown above. Then we have axial symmetry and can expand the potential in Legendre polynomials in each half.

$$\phi_j = A_0^{(j)} + B_0^{(j)}/r + \sum_{l=1}^{\infty} \left(A_l^{(j)} r^l + B_l^{(j)} r^{-l-1} \right) P_l(\cos\theta), \quad j = 1, 2.$$

At the conducting surfaces, $r = a$, $r = b$, the potentials are constant. Hence,

$$\begin{aligned} A_0^{(j)} + B_0^{(j)}/a &= V_a \\ A_0^{(j)} + B_0^{(j)}/b &= V_b \end{aligned}$$

while for $l \neq 0$

$$\begin{aligned} A_l^{(j)} a^l + B_l^{(j)} a^{-l-1} &= 0 \\ A_l^{(j)} b^l + B_l^{(j)} b^{-l-1} &= 0 \end{aligned}$$

For $l \neq 0$, all the coefficients are zero, and the potentials are

$$\phi_j = A_0^{(j)} + B_0^{(j)}/r\,.$$

On the interface we have $\phi_1(r) = \phi_2(r)$ for $a < r < b$. Hence the coefficients are equal, and we have everywhere

$$\phi = A + B/r\,.$$

The electric field is

$$\vec{E} = \frac{-B}{r^2}\hat{r}\,.$$

To find the value of B we apply the boundary condition for the normal component of the electric field. Let σ_0 and σ denote the free surface charge densities at $r = a$ on the two hemispheres of the conductor.

The boundary condition is

$$E_n(a) = -B/a^2 = \sigma_0/\epsilon_0 = \sigma/\epsilon\,,$$

which implies

$$\epsilon\sigma_0 = \epsilon_0\sigma\,.$$

Charge conservation gives

$$2\pi a^2(\sigma + \sigma_0) = Q\,.$$

The solution to these equations is

$$\begin{aligned} \sigma_0 &= \frac{\epsilon_0 Q}{(\epsilon + \epsilon_0)2\pi a^2} \\ \sigma &= \frac{\epsilon Q}{(\epsilon + \epsilon_0)2\pi a^2} \end{aligned}$$

and the field is

$$\vec{E} = \frac{-Q\hat{r}}{2\pi(\epsilon + \epsilon_0)r^2}\,.$$

(c) The total charge density at $r = a$ must be the same on both halves of the sphere since $\vec{E}$ is the same. Therefore,

$$\sigma + \sigma_P = \sigma_0$$

which gives

$$\sigma_P = \frac{-(\epsilon - \epsilon_0)Q}{(\epsilon + \epsilon_0)2\pi a^2}\,.$$

HW#7

Gang Wang

4.11 The Clausius-Mossotti relation reads,

$$\gamma_{mol} = \frac{3}{N}\left(\frac{k-1}{k+2}\right) \quad , \quad k = \frac{\varepsilon}{\varepsilon_0} \qquad ①$$

Since γ_{mol} is a constant for a given $\vec{E}$ field, by equation ①, $\frac{k-1}{k+2}$ should be proportional to the density of the substance N.

For air, one can get the following data from AIP Handbook

Pressure,	$\frac{\rho}{\rho_0}$	$\frac{k-1}{k+2}$	$\frac{\rho}{\rho_0} / \frac{k-1}{k+2}$
40 (atm)	36.72	7.21424×10^{-3}	5089.93
100 (atm)	91.61	1.7939×10^{-2}	5106.75

For Pentane.

pressure	ρ	$\frac{k-1}{k+2}$,	$\rho / \frac{k-1}{k+2}$	$k-1$.	$\rho / k-1$
1	0.613	0.21466	2.85568	0.82	0.7476
10^3	0.701	0.24242	2.891628	0.96	0.7302
4×10^3	0.796	0.27184	2.92819	1.12	0.7107
8×10^3	0.865	0.29245	2.95777	1.24	0.6976
12×10^5	0.907	0.30716	2.95285	1.33	0.68195

Both for pentane and for air, the C-M relation hold approximately

For air, the fractional variation is following.

$$\rho: \quad \frac{91.61-36.72}{36.72} = 149.5\%$$

$$k-1: \quad \frac{0.0548-0.0218}{0.0218} = 151.4\%$$

So k changes a little more apparent than the density ρ.

For pentane. the fractional variation is

$$\Delta\rho\% = \frac{\rho(p)-\rho(0)}{\rho(0)}$$

P	$\Delta(k-1)\%$	$\Delta\rho\ (\%)$
10^3	17.07%	14.36%
4×10^3	36.59%	29.85%
8×10^3	51.22%	41.11%
12×10^3	62.20%	47.96%

Comparing C-M relation with cruder relation $k-1 \propto \rho$. we find for the range considering. they both have a linear structure with ρ with similiar precesion