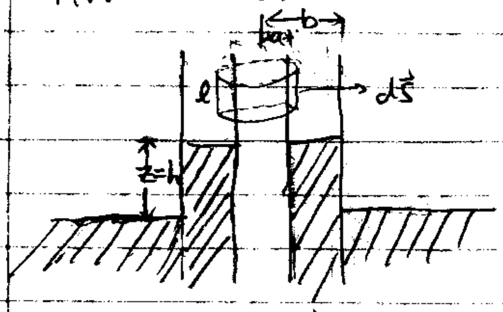
HW#8 solution.

1).




By symmetry, $\vec{E}$ should be independent of azimuthal angle and along radial direction.

From Gauss theorem, in the vaccum

$$\oint E\cdot d\vec{s} = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow E\,2\pi r l = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow E = \frac{\lambda}{2\pi r\varepsilon_0}, \quad \lambda = \frac{Q}{l}.$$

Since the voltage across the two conductor is V,

$$\int_a^b \vec{E}\cdot d\vec{r} = \int_a^b \frac{\lambda}{2\pi r\varepsilon_0}\,dr = V$$

$$\Rightarrow \lambda = \frac{2\pi\varepsilon_0 V}{\ln(\frac{a}{b})}$$

$$\Rightarrow E = \frac{V}{r\ln(\frac{b}{a})}$$

Clearly the electric field inside the medium is the same since $\varepsilon$ will be cancelled.

So, the electric energy contributed by the medium inside the conductor is

$$W_e = -\frac{1}{2}\int dV\,(\varepsilon-\varepsilon_0)E^2$$

$$= -\frac{1}{2}\int_0^z\int_0^{2\pi}\int_a^b (\varepsilon-\varepsilon_0)\left[\frac{V}{r\ln\frac{b}{a}}\right]^2 r\,dr\,d\theta\,dz$$

$$= -\frac{V^2}{\ln\frac{b}{a}}\pi(\varepsilon-\varepsilon_0)z \qquad ①$$

, where z is the height of the medium inside the conductor

If the system reaches equilibrium, the electric force will be equal to the gravity of the medium, i.e.

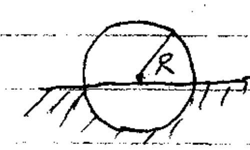
$$-\frac{\partial W_e}{\partial z} = \rho g \pi (b^2 - a^2) h \qquad ②$$

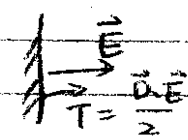
By ①, ②, one get

$$\chi_e = \frac{\varepsilon - \varepsilon_0}{\varepsilon_0} = \frac{\rho g (b^2 - a^2) h \ln(\frac{b}{a})}{\varepsilon_0 V^2}$$

2). Stress for fields perpendicular to a surface is

$$T = \frac{\vec{D} \cdot \vec{E}}{2}$$




From HW#7 (Jackson 4.10)

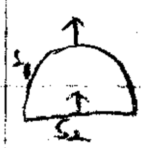
$$\vec{E} = \frac{Q}{2\pi(\varepsilon + \varepsilon_0) r^2} \hat{r} \qquad ①$$

If the system reaches equilibrium

$$\oint T \, d\vec{s} + mg\hat{z} + \frac{1}{2} \rho g \left(\frac{4}{3}\pi R^3\right) = 0 \qquad mg = \frac{4}{3}\pi R^3 \rho g f_0$$

$$\Rightarrow \int_{up} \frac{\varepsilon_0 E^2}{2} d\vec{s} + \int_{down} \frac{\varepsilon E^2}{2} d\vec{s} = -\rho g \frac{4}{3}\pi R^3 \left(\frac{1}{2} - f_0\right)$$

$$\Rightarrow \frac{1}{2} E^2 \left[\int_{up} \varepsilon_0 d\vec{s} + \int_{down} \varepsilon \, d\vec{s}\right] = -\rho g \frac{4}{3}\pi R^3 \left(\frac{1}{2} - f_0\right) \qquad ②$$




consider a closed surface as shown to the left

$$\oint \phi \, d\vec{s} = \int_V \nabla \phi \, dv$$

$$\Rightarrow \oint d\vec{s} = 0 \Rightarrow \int_{S_1} d\vec{s} = \int_{S_2} d\vec{s}$$

So, ② can be re write as

$$\frac{1}{2} E^2 (\varepsilon_0 - \varepsilon) \pi R^2 = -\rho g \frac{4}{3}\pi R^3 \left(\frac{1}{2} - f_0\right) \qquad ③$$

plug ① into ③, one get

$$Q = 2\pi(\varepsilon + \varepsilon_0) R^2 \left[\frac{8}{3} \rho g R \frac{\frac{1}{2} - f_0}{\varepsilon - \varepsilon_0}\right]^{\frac{1}{2}}$$