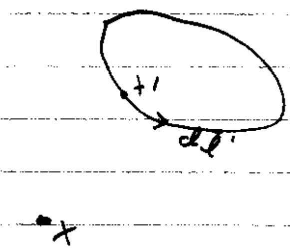
PHY 505

Homework No. 9

I 5.1



$$d\vec{B} = \frac{\mu_0 I}{4\pi}\, d\vec{l}' \times \frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}$$

$$= \frac{\mu_0 I}{4\pi}\, d\vec{l}' \times \nabla_{x'} \frac{1}{|\vec{x}-\vec{x}'|}$$

$$\vec{B}(\vec{x}) = \frac{\mu_0 I}{4\pi} \oint d\vec{l}' \times \nabla_{x'} \frac{1}{|\vec{x}-\vec{x}'|}$$

Variation of Gauss' Theorem:

$$\oint d\vec{l} \times \vec{A}(\vec{x}) = \int ds \left[\hat{n}_i \vec{\nabla} A_i - \hat{n}\, (\nabla \cdot A) \right]$$

Pf: consider for arbitrary constant vector $\vec{c}$

$$\oint d\vec{l} \times \vec{A}(x) \cdot c = \oint d\vec{l} \cdot (\vec{A}(x) \times \vec{c})$$

In this case $\displaystyle \oint d\vec{l}' \times \nabla'_x \frac{1}{|x-x'|} = \int d S' \left(n_i \nabla'_x \right) \nabla'_i \frac{1}{|x-x'|}$

$$- \hat{n}\, \nabla^2 \frac{1}{|\vec{x}-x'|} \Big)$$

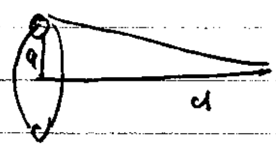
$$= \nabla_x \int \frac{d\vec{S}' \cdot (\vec{x}-\vec{x}')}{|x-x'|^3} \qquad \left(\nabla^2 \frac{1}{|x-x'|} = -4\pi\,\delta(x-x') \right.$$

but $\vec{x}$ is not on surface)

$$= \nabla_x \int d\Omega$$

$$\Rightarrow \quad \vec{B} = \frac{\mu_0}{4\pi} \nabla \Omega$$

2. Jackson 6.3

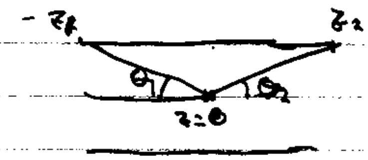


For a loop of current - on axis field

$$B_z = \frac{\mu_0 I}{2} \frac{a^2}{\{a^2+d^2\}^{3/2}}$$

For total field of coil

$$B_z = \sum_{\text{coils}} \frac{\mu_0 I}{2} \frac{a^2}{[a^2+z_i^2]^{3/2}}$$



$$z_1 + z_2 = NL$$

In limit $NL \to \infty$,

$$\sum_i f(z_i) \to N \int f(z)\,dz$$

$$B_z = \frac{\mu_0 N I}{2} \int_{-z_1}^{z_2} \frac{a^2\,dz}{[a^2+z^2]^{3/2}}$$

Split integral into $\int_{-z_1}^{0} + \int_{0}^{z_1}$

For $z>0$, let $z = a \operatorname{ctn}\theta$

$$\int_0^{z_2} \frac{a^2\,dz}{\{a^2+z^2\}^{3/2}} = -\int_{\pi/2}^{\theta_2} \sin\theta\,d\theta = \cos\theta_2$$

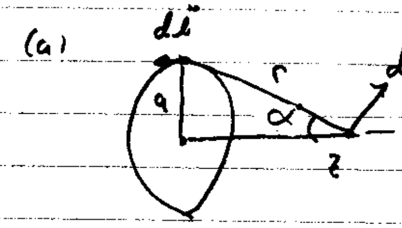
For $z<0$, let $z = -a \operatorname{ctn}\theta$

$$\int_{-z_1}^{0} \frac{a^2\,dz}{[a^2+z^2]^{3/2}} = \int_{\theta_1}^{\pi/2} \sin\theta\,d\theta = \cos\theta_1$$

Here $B_z = \frac{\mu_0 N I}{2}\{\cos\theta_1 + \cos\theta_2\}$

Jackson 5.7 - Helmholtz Coils.

(a)



Let current loop lie in x-y plane with z-axis through center of circle. Use cylindrical coordinates ρ, θ, z.

(i) By axial symmetry $\quad \vec{B}(\rho,\theta,z) = \vec{B}(\rho,\theta+\delta,z)$

$\Rightarrow$ no θ dependence

(ii) Consider a circle parallel to x-y plane with center on z-axis at pt z with radius ρ. No current passes through loop. Therefore,

$$0 = \oint \vec{B}\cdot d\vec{l} = 2\pi\rho\, B_\phi(\rho,z) \Rightarrow B_\phi = 0$$

~~Ans~~ $$\vec{B} = \hat{\rho}\, B_\rho(\rho,z) + \hat{z}\, B_z(\rho,z)$$

(iii) It also follows by axial symmetry that on the z-axis, $\vec{B} = \hat{z}\, B_z(0,z)$

(i.e. $B_\rho(0,z) = 0$.)

$$\therefore \quad (dB)_z = \left(\frac{\mu_0 I}{4\pi}\right)\frac{dl}{r^2}\sin\alpha$$

$$= \left(\frac{\mu_0 I}{4\pi}\right) dl\, \frac{a}{r^3}$$

$$\Rightarrow \quad B_z = \frac{\mu_0 I}{2}\frac{a^2}{r^3}, \quad r = \{a^2+z^2\}^{1/2}$$

(b) With the 2 coils arranged as described in the text, we have:

$$B_z(0,z) = \frac{\mu_0 I a^2}{2}\left[\left(a^2 + (z-b/2)^2\right)^{-3/2} + \left(a^2+(z+b/2)^2\right)^{-3/2}\right] \qquad (A)$$

We want to expand in powers of z/d^2, $d^2 = a^2 + b^2/4$, so we write

$$B_z(0,z) = \frac{\mu_0 I a^2}{2d^3}\left\{\left\{1 - \frac{bz}{d^2} + \frac{(a^2+b^2/4)z^2}{d^4}\right\}^{-3/2} + \left\{1 + \frac{bz}{d^2} + \frac{(a^2+b^2/4)z^2}{d^4}\right\}^{-3/2}\right\} \qquad (B)$$

When we expand these terms odd powers of z/d^2 cancel and even powers add up. To get the required accuracy of the formula in Jackson, we need a Taylor series up to 4th order, i.e.,

$$\{1+x\}^{-3/2} = 1 - \tfrac{3}{2}x + \frac{1}{2!}\left(\frac{3\cdot5}{2^2}\right)x^2 - \frac{1}{3!}\left(\frac{3\cdot5\cdot7}{2^3}\right) + \frac{1}{4!}\left(\frac{3\cdot5\cdot7\cdot9}{2^4}\right) + O(x^5)$$

The algebra yields:

$$B_z(0,z) = \left(\frac{\mu_0 I a^2}{d^3}\right)\left\{1 + \frac{3(b^2-a^2)z^2}{d^4} + \frac{15(b^4-6b^2a^2+2a^4)z^4}{d^8} + O\left(\left(\frac{z^2}{d^4}\right)^3\right)\right\}$$

(c) Near the origin, we can expand B_z & B_ρ up to 2nd order in the coordinates as.

$$B_z(\rho,z) = \sigma_0 + \sigma_2 z^2 + a\rho + b\rho z + c\rho^2$$

$$B_\rho(\rho,z) = d\rho + e\rho z + f\rho^2$$

σ_0 & σ_2 are given by part (b). The remaining constants, $a, \ldots, f$, can be determined by applying the field equations
$\nabla \times \vec{B} = 0$; $\nabla \cdot \vec{B} = 0$

In this case, they become:

$$\frac{\partial B_\rho}{\partial z} - \frac{\partial B_z}{\partial \rho} = 0 \quad ; \quad \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\rho) + \frac{\partial B_z}{\partial z} = 0$$

The 1st equation gives: $e\rho = a + bz + 2c\rho$
Hence, $a = 0$, $b = 0$, $\underline{e = 2c}$.

The 2nd equation gives: $2d + 2ez + 3f\rho = -(2\sigma_2 z + b\rho)$

Hence $d = 0$, $3f = b = 0$, $e = -\sigma_2$ $(= 2c$

Putting this together, we have

$$B_z(\rho,z) = \sigma_0 + \sigma_2(z^2 - \rho^2/2)$$

$$B_\rho(\rho,z) = -\sigma_2 \rho z$$

Note: If we use the on-axis information $B_z(0,z) = \sigma_0 + \sigma_2 z^2 + \sigma_4 z^4$, we can determine $\vec{B}(\rho,z)$ completely to 4th order by the same method.
Results: $B_z(\rho,z) = \sigma_0 + \sigma_2(z^2 - \rho^2/2) + \sigma_4(z^4 - 3\rho^2 z^2 + \tfrac{3}{8}\rho^4)$

$$B_\rho(\rho,z) = -\sigma_2 \rho z + \sigma_4(\tfrac{3}{2}\rho^3 z - 2\rho z^3).$$

(d) For an expansion a large z, we can write eqn. (A) as

$$B_z(0,z) = \frac{\mu_0 I}{2|z|^3}\left\{\left(1+\frac{a^2+b^2/4}{|z|^2}-\frac{b}{|z|}\right)^{-3/2} + \left(1+\frac{a^2+b^2/4}{|z|^2}+\frac{b}{|z|}\right)^{-3/2}\right\}$$

Comparing to eqn. (B), we see that the eqn above is obtained from (B) by the substitution $d \rightarrow |z|$.

(e) For $b = a$, the expression for the axial field becomes

$$B_z = \left(\frac{\mu_0 I a^2}{d^3}\right)\left[1 - \frac{144}{125}\left(\frac{|z|}{a}\right)^4 + \cdots\right]$$

(i) For an accuracy of 1 part in 10^4, we require

$$\left(\frac{144}{125}\right)\left(\frac{|z|}{a}\right)^4 = 10^{-4}$$

or $\frac{|z|}{a} \lesssim 10^{-1}$

(ii) For an accuracy of one part in 10^{-2}

we require $\frac{|z|}{a} \lesssim 10^{-1/2} \approx 0.3$