PHY505 Classical Electrodynamics
Electrostatic Green Functions

The electrostatic Green Function satisfies the equation

$$\nabla^2 G(\vec{x}_1, \vec{x}_2) = -\delta^{(3)}(\vec{x}_1 - \vec{x}_2)$$

in the presence of conducting surfaces $\{S_{a_j}\}$ and $\{S_{b_k}\}$ where

a) the Green Function vanishes on the a_j surfaces

$$G(\vec{x}_1, \vec{x}_2) = 0, \quad \vec{x}_1 \in S_{a_j}$$

b) the total charge on the b_k surfaces is zero,

$$\oint_{S_{b_k}} \frac{\partial}{\partial n} G(\vec{x}, \vec{x}_2)\, dS = 0$$

Note that $G(\vec{x}_1, \vec{x}_2)$ must be constant when $\vec{x}_1$ is on S_{b_k} since it is the surface of a conductor.

To find G we may first solve with $G = 0$ on the type a surfaces, and $G = V_{b_k}$ on S_{b_k}. At this point G is a function of the parameters $\{V_{b_k}\}$, which have not yet been determined. To find them we use the conditions of b) to fix their values. Note that the value of $\{V_{b_k}\}$ depend on the position of the source $\vec{x}_2$.

The equations for the V_{b_k} derived from the vanishing charge conditions are in fact linear inhomogeneous equations. This follows from the linearity of the Poisson equation and the superposition principle. To see this explicitly we can write G in terms of G^D, the Dirichlet Green function which vanishes on both the type a and type b surfaces. Using Green's Theorem with $\phi = G(\vec{x}, \vec{x}_2)$ and $\chi = G^D(\vec{x}, \vec{x}_1)$ gives

$$G(\vec{x}_1, \vec{x}_2) = G^D(\vec{x}_1, \vec{x}_2) - \sum_{k=1}^{n} V_{b_k}(\vec{x}_2) \oint_{S_{b_k}} \frac{\partial}{\partial n} G^D(\vec{x}, \vec{x}_1)\, dS$$

Now impose the vanishing charge conditions to obtain

$$\sum_k M_{jk} V_{b_k}(\vec{x}_2) = Q_j^D(\vec{x}_2)$$

where

$$Q_j^D(\vec{x}_2) = \oint_{S_{b_j}} \frac{\partial}{\partial n} G^D(\vec{x}, \vec{x}_2)\, dS$$

and

$$M_{jk} = \oint_{S_{b_j}} \oint_{S_{b_k}} \frac{\partial}{\partial n_j} \frac{\partial}{\partial n_k} G^D(\vec{x}_j, \vec{x}_k)\, dS_j\, dS_k$$

From the formulas above it follows that the coefficient matrix M is symmetric, $M_{jk} = M_{kj}$. M is independent of the position of the δ–function source and depends only on the geometry of the conductors.

Substituting the solution for V_{b_j} we have the explicit result

$$G(\vec{x}_1, \vec{x}_2) = G^D(\vec{x}_1, \vec{x}_2) - \sum_{j,k=1}^{n} Q_j^D(\vec{x}_1) M jk^{-1} Q_j^D(\vec{x}_2)\,.$$