PHY 308 INTRODUCTION TO QUANTUM MECHANICS SPRING 2005

FINAL EXAMINATION Friday 13 May 2005 8-1030 am

NAME: ____

This is a closed-book examination. Numerical results need only have one-place accuracy, so that a calculator is not required (but is permitted).

Answers, INCLUDING ALL STEPS, should be presented on front and back of each examination sheet. Given the limited space, you may wish to do the work on scrap paper and then fill it in. Recommended time allocations are indicated for each question, and credit will be proportional to these times. The total exam contributes 40% to the course grade.

Basic constants: h = 6.6×10^{-34} Joule-s, c = 3×10^8 m/s, $m_{\text{electron}} = 0.91 \times 10^{-30}$ kg

1. (15 min) For light with wavelength λ, what is the momentum of each photon? How does this relate to the energy $E = h\nu$? Does this agree with Einstein's special relativity formula $E = \sqrt{(pc)^2 + (mc^2)^2}$? What does it say about the rest mass of a photon? Check the Einstein formula for a particle with nonzero rest mass moving slowly ("nonrelativistically") by expanding the square root through first order in the small quantity $(p/mc)^2$.

Solution: The de Broglie formula is $p = h/\lambda$. This was derived in class by considering the momentum density in the electromagnetic field. That in turn implies $E = h\nu = hc/\lambda = pc$, which agrees with the formula $E = \sqrt{(pc)^2 + (mc^2)^2}$, provided $m = 0$. For nonzero m, and $p << mc$ one may expand the square root as follows:

$$mc^2\sqrt{(1 + (p/mc)^2)} \approx mc^2(1 + (p/mc)^2/2) = mc^2 + p^2/2m \quad ,$$

which is just the rest-energy plus the Newtonian expression for kinetic energy.

2. (15 min) Let $H/\hbar = \pi\sigma_3/T$, and $\psi(t=0) = (1/\sqrt{2})\begin{pmatrix}1\\1\end{pmatrix}$. Find $\psi(t=T/2)$ and $\psi(t=T)$. Are either or both of these different physical states from $\psi(t=0)$? Explain your reasoning. $\left[\text{Note: } \sigma_3 \equiv \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\right]$

Solution:

$$\psi(t=T/2) = (1/\sqrt{2})\begin{pmatrix}e^{-i\pi/2}\\e^{i\pi/2}\end{pmatrix} = (i/\sqrt{2})\begin{pmatrix}-1\\1\end{pmatrix} \quad ,$$

and

$$\psi(t=T/2) = (1/\sqrt{2})\begin{pmatrix}e^{-i\pi}\\e^{i\pi}\end{pmatrix} = (-1/\sqrt{2})\begin{pmatrix}1\\1\end{pmatrix} \quad ,$$

The first expression is orthogonal to $\psi(t=0)$, and so clearly is a different physical state. The second expression differs from $\psi(t=0)$ only by a multiplicative phase factor (-1), and because that phase is unobservable this is the same physical state as $\psi(t=0)$. It corresponds to a rotation by 2π about the z axis, and such a rotation leaves all observables unchanged.

3. Consider the Schrödinger equation with dimensions scaled out, $-\partial_x^2\psi + x^2\psi = 2E\psi$. Here x and E are treated as pure numbers, rather than having dimensions of length and energy. Show that the choice $\psi = e^{-x^2/2}$ solves this equation for a special value of E, and determine that value.

Solution:

$$\partial_x e^{-x^2/2} = -xe^{-x^2/2},$$

and

$$\partial_x(-xe^{-x^2/2}) = x^2e^{-x^2/2} - e^{-x^2/2} \quad .$$

This gives for the dimensionless Schrödinger equation

$$(-x^2 + 1 + x^2)e^{-x^2/2} = e^{-x^2/2} = 2Ee^{-x^2/2} \quad ,$$

or

$$1 = 2E \quad .$$

Therefore we get

$$E = 1/2 \quad .$$

4. (15 min) Find the wave function corresponding to the ground state of the hydrogen atom, meaning a wave function depending only on r, of the form $e^{-\kappa r}$. You do not need to get the normalization of the wave function. The starting point is the Schrödinger equation,

$$-\hbar^2(\psi''(r) + 2(\psi'(r)/r))/2m - e^2\psi(r)/r = E\psi(r) \quad .$$

Relate κ to the Bohr radius $a = \hbar^2/me^2$, and relate E to κ and m.

Solution:

$$\psi'(r) = - - \kappa e^{-\kappa r} \quad ,$$

giving

$$-(\psi''(r) + 2(\psi'(r)/r)) = (-\kappa^2 + 2\kappa/r)e^{-\kappa r} \quad .$$

Substituting into the Schrödinger equation multiplied by $2m/\hbar^2$ *gives*

$$((-\kappa^2 + 2\kappa/r)e^{-\kappa r} - (2/ar + 2mE/\hbar^2)e^{-\kappa r} \quad ,$$

or $\kappa = 1/a$ *and* $E = -\hbar^2\kappa^2/2m = -e^2/2a$.

5. (Essay question, 30 min) Describe the magic numbers for closed shells in atoms, on the assumption that the forces between electrons can be neglected. What are the qualitative changes when electron screening of the positive nuclear charge is taken into account?

Ignoring electron-electron interactions gives independent electrons around a nucleus with charge Ze. A neutral atom has Z electrons, with levels (obtained from the Schrödinger equation) filled from the lowest up. Principal quantum number n gives $E_n = -Z^2e^2/2an^2$, with $a = \hbar^2/me^2$ the hydrogen Bohr radius. For each n the number of different space wave functions is n^2, and by the Pauli principle the number of electrons to completely fill such a level is $N_n = 2n^2$, one spin-up and one spin-down electron for each space wave function. This gives magic numbers for chemically inert atoms with filled shells 2, 2+8=10, 2+8+18=28, etc. Because inner electrons shield part of the nuclear charge from electrons farther out, the actual magic numbers increase more slowly – wave functions far out have less binding because of the charge shielding. Therefore the actual magic numbers are $Z = 2, 10, 18$, etc. In the third row of the periodic table $n = 3, \ell = 2$ states are so far out that they are not bound until after $n = 4, \ell = 0, 1$ states have been filled, which brings one to the next shell, so the $n = 3$ electrons are chemically unimportant because they don't stick out as far as $n = 4$.

6. (Essay question, 30 min) Imagine a solution of the Schrödinger equation with energy E. Suppose that the potential in the equation is $V = 0$ for $x < 0$ and $V = V_0 > E$ for $x > 0$. Discuss the different results in classical and quantum physics for the possible presence of a particle in the region $x > 0$.

In classical physics, a particle must always have non-negative kinetic energy. Because the kinetic energy is $p^2/2m = E - V$, in the region to the right of $x = 0$ the kinetic energy would be negative, so it is impossible for the particle to be found there. Therefore this is called a 'classically forbidden zone'. On the other hand, in quantum mechanics $p^2 = -\hbar^2 \partial_x^2 < 0$ just means that the wave function is exponentially increasing or decreasing. Given a beam coming from the left, it makes no sense to have an exponentially increasing probability density on the right, so the only option is exponential decrease. Therefore, instead of vanishing to the right of $x = 0$, the probability decreases exponentially as $e^{-2\sqrt{(2m(E-V_0)}\ x/\hbar}$. This is an example of an uncertainty-principle effect: The finite energy barrier cannot keep particles completely out of the classically forbidden zone.

7. (30 min, EXTRA CREDIT FOR THOSE WHO STAY THE FULL EXAM TIME, OTHERWISE COUNTS FOR 1/5 OF THE EXAM)

A uniform beam of objects (meaning fixed number per unit area per unit time) is incident with $v_x = 100$ m/s on a wall in the $x = 0$ plane. There is a slit in the wall, 1 cm wide, centered on the z axis ($x = y = 0$).

For each of the following three cases, describe as quantitatively as possible the pattern seen on a second wall, 100 m in back of the first, with detectors which can determine position to an accuracy of 1 mm. Absolute normalization is neither required nor possible with the information given.

a. The objects are 30 kg standard-size bicycles.

Solution. Obviously the bicycles cannot get through, so there is negligible intensity everywhere on the second wall. For nitpickers, one may imagine that pieces of the bicycles come off when they crash into the wall, passing through the slit and landing somewhere on the second wall, but the intensity of intact bicycles still is exactly zero! Some students observed that the bicycles might have crashed through the wall, and that in principle is an option. However, my interpretation of the question is that the wall 'is', and therefore remains intact throughout the period when the objects are incident on it.

b. The objects are miniature dice, 1 μm edges, with mass 10^{-15} kg.

Solution. As will be shown below, with a velocity of 100 m/s these objects have far too high momentum for any wave diffraction effects to be visible on the second screen where resolved features must be separated from others by at least 1 mm. Consequently, we can use straight-line particle trajectories, yielding a uniform intensity occupying a 1 cm wide strip centered on the line $y = 0$ *in the plane of the second wall. Outside that strip the intensity is negligible, because it could come only from scattering on the edge of the slit, and for such small objects the probability of coming close enough to either edge to have even the possibility of scattering is (1 μ m / 1 cm* $= 10^{-4}$*).*

c. The objects are electrons.

Solution. Now the mass is 0.91$\times 10^{-30}$ *kg, and for velocity of 100 m/s this gives momentum* p = *0.91*$\times 10^{-28}$ *kg-m/s, and wavelength* $\lambda = h/p$ = *10^{-5} m (to one-place accuracy). The angle* θ *for the first minimum in the diffraction pattern of the slit is given by* $\lambda/2 = \mathrm{w}\sin(\theta)/2$, *where* w *is the width of the slit (1 cm). Using the small-angle approximation* $\theta(\mathrm{rad}) \approx \sin\theta \approx \tan\theta$, *we have* $\tan\theta \approx 10^{-5}$ *m/ 10^{-2} m = 10^{-3}. At the second wall this means the first minimum comes at* $10^{-3} \times 100$ *m = 10 cm away from the line* $y = 0$. *With 1 mm resolution this minimum should be easily visible.*

What is the largest mass the objects could have, such that a diffraction pattern still could be discerned on the second wall?

Solution. From the previous calculation for an electron, we see that the distance from the central maximum to the first minimum is 10 cm, 100 times the resolution of 1 mm. Therefore, for a mass 100 times greater than the electron mass the distance would be just equal to the resolution, meaning a mass (to 1 place accuracy) of 10^{-28} kg.