Physics 308 Introduction to Quantum Mechanics Spring 2005

Homework 11, Due AT BEGINNING OF CLASS, Wednesday 20 April

Orbital angular momentum $\ell = 1$

To simplify discussion in this homework, use units with $\hbar = 1$. Then the eigenvalues of the orbital angular momentum are $\ell(\ell+1)$, and for a given ℓ a 'magnetic quantum number', or eigenstate of L_z, is $m = \ell, \ell-1, ..., -\ell$.

1. Consider the function $\psi(r, \theta, \phi) = N(x+iy) = Nr\sin\theta e^{i\phi}$. For $r = 1$, find the normalization factor N (which you should assume real and positive), such that the integral $\int d\Omega|\psi|^2 \equiv \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi|\psi|^2 = 1$. [Hint: If we choose $w \equiv \cos\theta$, then $\int_0^\pi d\theta = \int_{-1}^1 dw$. Also, $\sin^2\theta = 1 - \cos^2\theta$.]

2. The operator $L_\pm = L_x \pm iL_y$ has matrix elements $\langle \ell, m \pm 1|L_\pm|m\rangle = \sqrt{(\ell(\ell+1) - m(m \pm 1))}$. Find $L_-\psi$ acting on the normalized wave function of problem 1, which corresponds to $\ell = 1, m = 1$, and deduce the normalized wave function for $\ell = 1, m = 0$ (being especially careful about the sign of the wave function).

3. Repeat the process once more to get the normalized wave function (being especially careful about the sign of the wave function) for $\ell = 1, m = -1$.

Spin angular momentum $s = 1/2$

4. Represent spin up in the z direction by a spinor or two-component vector with upper component 1 and lower component 0. From the general analysis we know that a case with only two spin values must have eigenvalues of spin $m_s = \pm 1/2$ (because the upper one is plus s and the lower one is $-s$, and the difference $s - (-s)$ is 1). Clearly this means that the matrix whose eigenvalues are m_s must be $\sigma_3/2$. Find the expectation value, for the state with $s_z = +1$, of the spin in the x direction, given by the matrix $\sigma_x/2$.

5. Find the expectation values of s_x^2 and s_y^2. Comment on the contrast with classical physics, where if a spin vector is pointing in a certain direction with its maximum possible value, by definition the spin components in the perpendicular directions vanish.