Physics 308 Introduction to Quantum Mechanics Spring 2005

Homework 13 (Extra Credit), Answers

1. Describe how the two assumptions for photons $E = h\nu$ and $\rho = [\epsilon_0 \vec{E}^2 + \vec{B}^2/\mu_0]/8\pi h\nu$, both at least implicit in Einstein's 1905 paper on particle phenomena for light, give single-particle quantum mechanics and so a natural introduction to all quantum mechanics. [Here ρ is the number of photons per unit volume.]

In his 1905 paper on particle aspects of light, Einstein stated that each light particle (today called a 'photon') carries an energy $E = \mathrm{h}\nu$, where h is Planck's constant and ν is the frequency of the light. He also said that the rate at which photons knock out electrons from a metal is proportional to the intensity of the light beam. A straightforward way to interpret this is that photon intensity is given by the power intensity of the electromagnetic field, divided by the energy per photon. Consequently, the field energy density should determine the density of photons. This implies that photons don't necessarily travel like particles, but instead are 'told' where to go by the wave – one can't predict exactly where a photon will go, but only give probabilities depending on the wave intensity.

2. Describe the magic numbers for closed shells in atoms, on the assumption that the forces between electrons can be neglected. What are the qualitative changes when electron screening of the positive nuclear charge is taken into account?

Ignoring electron-electron interactions gives independent electrons around a nucleus with charge Ze. A neutral atom has Z electrons, with levels (obtained from the Schrödinger equation) filled from the lowest up. Principal quantum number n gives $E_n = -Z^2e^2/2an^2$, with $a = \hbar^2/me^2$ the hydrogen Bohr radius. For each n the number of different space wave functions is n^2, and by the Pauli principle the number of electrons to completely fill such a level is $N_n = 2n^2$, one spin-up and one spin-down electron for each space wave function. This gives magic numbers for chemically inert atoms with filled shells 2, 2+8=10, 2+8+18=28, etc. Because inner electrons shield part of the nuclear charge from electrons farther out, the actual magic numbers increase more slowly – wave functions far out have less binding because of the charge shielding. Therefore the actual magic numbers are $Z = 2, 10, 18$, etc. In the third row of the periodic table $n = 3, \ell = 2$ states are so far out that they are not bound until after $n = 4, \ell = 0, 1$ states have been filled, which brings one to the next shell, so the $n = 3$ electrons are chemically unimportant because they don't stick out as far as $n = 4$.

3. Describe the closest quantum-mechanical harmonic-oscillator solution to the behavior of a classical harmonic oscillator, whose coordinate goes back and forth with sinusoidal time dependence. What is the crucial difference between this solution and the classical one? When is that difference unimportant?

The ground-state wave function for a harmonic oscillator is a gaussian function centered on the equilibrium point of the oscillator, with the minimum value for the uncertainty product $\delta x \delta p$ allowed by the uncertainty principle. If at the start this wave function is centered about a point a distance X from the equilibrium point and released, it will swing back and forth between X and $-X$ just like a classical mass at the end of a spring, and at the same frequency as for the classical harmonic oscillator. The difference is that we cannot describe the quantum wave as a point, because it has a size. Therefore the classical description of the motion will be accurate as long as X is much greater than the width $\delta x \sim \sqrt{(h/\sqrt{(km)}}$.

4. Two basic ideas in quantum mechanics are the notion that phase space is quantized, that is, a finite area h in (x, p) space is needed to accommodate a possible state of the system, and the uncertainty principle which says that the product $\delta x \delta p$ must be no smaller than $\hbar/2$. Explain why it is natural that one involves h and the other $\hbar$.

The most compact shape for a certain area is a circle, or if the scales are different in x and p directions, an ellipse. The area of an ellipse with length $2X$ in the x direction and $2P$ in the p direction is πXP, which is easily checked by scaling out the dimensions to get an integral over the unit circle. On the other hand, assuming the ellipse is centered on the origin, δx^2 can be obtained by integrating x^2 over the ellipse and dividing by the area of the ellipse. This gives $\delta x^2 = X^2/4$, and $\delta p^2 = P^2/4$. Then we have $\delta x \delta p = XP/4$. Setting the area πXP equal to h, we get $XP/4 = \mathrm{h}/4\pi = \hbar/2$. Thus a uniform distribution in phase space with total area h automatically gives the minimum possible value for the uncertainty product.

5. Imagine a solution of the Schrödinger equation with energy E. Suppose that the potential in the equation is $V = 0$ for $x < 0$ and $V = V_0 > E$ for $x > 0$. Discuss the different results in classical and quantum physics for the possible presence of a particle in the region $x > 0$.

In classical physics, a particle must always have non-negative kinetic energy. Because the kinetic energy is $p^2/2m = E - V$, in the region to the right of $x = 0$ the kinetic energy would be negative, so it is impossible for the particle to be found there. Therefore this is called a 'classically forbidden zone'. On the other hand, in quantum mechanics $p^2 = -\hbar^2 \partial_x^2 < 0$ just means that the wave function is exponentially increasing or decreasing. Given a beam coming from the left, it makes no sense to have an exponentially increasing probability density on the right, so the only option is exponential decrease. Therefore, instead of vanishing to the right of $x = 0$, the probability decreases exponentially as $e^{-2\sqrt{(2m(E-V_0)}\ x/\hbar}$. This is an example of an uncertainty-principle effect: The finite energy barrier cannot keep particles completely out of the classically forbidden zone.