PHY 308 INTRODUCTION TO QUANTUM MECHANICS SPRING 2005

Homework 7 (Due Wednesday 16 March 2005)

Hermitian vs Self-Adjoint Operators; Quantum Harmonic Oscillator

1. The momentum operator p acting on a wave function in one space dimension is $p\psi(x) = -i\hbar\partial_x\psi(x)$. Consider the wave functions of the infinite square well, which vanish at $x = 0$ and $x = a$ and are nonzero only on the interval between these two points. Show that the operator p is Hermitian when acting on these wave functions, i.e., $\langle\phi|p|\psi\rangle = \langle\psi|p|\phi\rangle^*$, or $\int dx\phi^*(x)(-i\hbar\partial_x)\psi(x) = [\int dx\psi^*(x)(-i\hbar\partial_x)\phi(x)]^*$.

[Hint: For functions vanishing at the endpoints, $\int dx(f(x)\partial_x g(x) + g(x)\partial_x f(x)) = 0$.]

2. Now suppose a wave function obeys the condition $p|\psi\rangle = \hbar k|\psi\rangle$, for some real number k. Find the form of the wave function $\psi(x)$. By 'form' is meant that you need not specify any particular constant coefficient to normalize the wave function.

3. Can the wave function of definite momentum found in Prob. 2 be made to satisfy the boundary conditions at $x = 0$ and $x = a$ of Prob. 1? The condition of self-adjointness is that the operator has a complete set of orthogonal, normalized eigenstates in the space of allowed wave functions. Is this possible for p when the boundary conditions of Prob. 1 are required?

4. The Hamiltonian for the simple harmonic oscillator [Griffiths Ch. 2] is $H = p^2/2m + kx^2/2$. Rescale this Hamiltonian as $H = \hbar\omega\tilde{H}$, where $\tilde{H} = (1/2)(-\partial_\xi^2 + \xi^2)$. What are the values of the circular frequency ω and the ratio ξ/x?

5. Let an eigenfunction of $\tilde{H}$ be $\psi_2(\xi) = (1+\alpha\xi^2)e^{-\xi^2/2}$. Find the **nonzero** value of α for which this function satisfies the equation $\tilde{H}\psi = e\psi$, and the corresponding value of the dimensionless energy e.

[Hint: The quickest way to find α may be to insist that this wave function is orthogonal to the ground-state wave function you studied before, $e^{-\xi^2/2}$.]