PHY 308 INTRODUCTION TO QUANTUM MECHANICS SPRING 2005

MIDTERM EXAMINATION Wednesday 9 March 2005

NAME: ______________________________

This is a closed-book examination. Numerical results need only have one-place accuracy, so that a calculator is not required (but is permitted). Answers, INCLUDING ALL STEPS, should be presented on the (two-sided) examination sheet. Given the limited space, you may wish to do the work on scrap paper and then fill it in.

Basic constants: h = 6.6×10^{-34} Joule-s, c = 3×10^8 m/s

1. Show that the 2 by 2 matrix $U = e^{-iHt/\hbar}$, where $H = h_1\sigma_1 + h_2\sigma_2 + h_3\sigma_3$ (all h_i real), is unitary: $U^\dagger U = 1$.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Solution A: If $f(\mathcal{O})$ is a function of an operator, then $f^*(\mathcal{O}^{\dagger)} = [f(\mathcal{O})]^\dagger$. Here the * means reversing signs of explicit factors i in f, and † means adjoint of an operator. In the present case, because the h_i are real and the σ_i are Hermitian finite matrices, therefore also self-adjoint, we have $H^\dagger = H$, and therefore $U^\dagger = e^{iHt}$. The product $U^\dagger U = e^{iHt}e^{-iHt}$. Because H clearly commutes with itself, this product also can be written $e^{iHt-iHt} = e^0 = 1$, which is the meaning of the statement that U is unitary.

Solution B: This goes the same way as the previous one, but then one computes $d[U^\dagger U]/dt = U^\dagger(iH - iH)U = 0$. As $U(t=0) = 1$, it follows that $U^\dagger U = 1$ at $t = 0$, and therefore $U^\dagger U = 1$ at all later times.

Solution C: There exists a set of two eigenvectors with eigenvalues of U with eigenvalues that are just phase factors. Thus in the basis of those eigenvectors U is a diagonal matrix with phase factors as the two entries, and $U\dagger$ is the matrix with complex conjugate phase factors, so that the product of the two is the unit matrix in this basis, and therefore is the unit matrix when acting on any vector.

2. For ultraviolet light with wavelength $\lambda = 10^{-7}$ m, what is the momentum of each photon?* How does this relate to the energy $E = h\nu$? Does this agree with Einstein's special relativity formula $E = \sqrt{(pc)^2 + (mc^2)^2}$? What does it say about the rest mass of a photon? Check the Einstein formula for a particle with nonzero rest mass moving slowly ("nonrelativistically") by expanding it through first order in the small quantity $(p/mc)^2$.

[*This first question is the ONLY part of the problem for which a numerical value is required.]

(a) $p = h/\lambda = 7 \times 10^{-27}$ *kg-m/s or N-s or J-s/m.*

(b) $e = h\nu = hc/\lambda = pc$.

(c) This agrees with $E = \sqrt{(pc)^2 + (mc^2)^2}$ *provided the rest mass* m *is zero.*

(d) For small p/mc *we may use* $\sqrt{1+\epsilon} \approx 1 + \epsilon/2$ *to get* $E \approx mc^2(1 + (1/2)(p/mc)^2) = mc^2 + p^2/2m$. *This us just the rest energy plus the usual nonrelativistic kinetic energy, confirming that Einstein's formula has the correct Newtonian limit.*

3. Consider the Schrödinger equation with dimensions scaled out, $-\partial_x^2\psi + x^2\psi = 2E\psi$. Here x and E are treated as pure numbers, rather than having dimensions of length and energy. Show that the choice $\psi = e^{-x^2/2}$ solves this equation for a special value of E, and determine that value.

We have $-\partial_x^2\psi = -\partial_x x e^{-x^2/2} = (x^2 - 1)e^{-x^2/2}$. *Therefore* $-\partial_x^2\psi + x^2\psi = (-x^2 + 1 + x^2)\psi \equiv 2E\psi$, *or* $E = 1/2$. *Note that we didn't put in factors to normalize* ψ *so that* $\int dx|\psi|^2 = 1$, *but that doesn't matter because a constant normalization coefficient would factor out in the linear Schrödinger equation.*