Much Ado about $N = 2$

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Supergravity at 25 – Stony Brook 2001

$N=2$ supergravity : Ferrara & Van Nieuwenhuizen

reports at Supergravity '79 Stony Brook

emphasis on auxiliary fields &
off-shell structure

Why $N=2$? more complete unifications
ultraviolet finiteness
no off-shellness beyond 8 supercharges

matter – hypermultiplets → hyperKähler
Quaternion-Kähler

gauge theory – vector multiplets → special geometry
– tensor multiplets

supergravity

Large variety of { theories
applications

effective actions for

type-II string compactification on CY_3
heterotic string compactification on $K3 \times T^2$
Seiberg-Witten theory
fixed point behaviour, black hole entropy
R^2 terms etc.

Participants of the Workshop on Supergravity
Stony Brook, 27-29 September, 1979

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CONTENTS

methods

- superconformal multiplet calculus (this talk)

  $\Longleftrightarrow$ Quotients

  insisted (originally) on off-shellness (not necessary!)

  $\rightarrow$ restricted class

- harmonic superspace

  infinite # degrees of freedom

  quantum corrections

- more straightforward approaches or geometric ones (eg group manifold)

Early work

Breitenlohner & Sohnius: Quaternion-kähler quotients (in susy setting)

Bagger & Witten: imposed mathematical structure (still unresolved issues);

Alvarez-Gaumé & Freedman: target space geometry

Hitchin, Karlhede, Lindström, Roček: quotients

Early 90's

Swann, Galicki

- various quotients commute
- generalized superconformal quotient (i.e. relaxing off-shellness)
- mathematical structure of hyperkähler cone $\rightarrow$ quaternion-kähler space

1. SUPERSYMMETRY

Field theories with 8 supersymmetries

$N=2 \quad D=4$

$N=1 \quad D=5,6$

"R-symmetry" group (relativistic internal automorphism group of susy algebra)

$USp(2,\mathbb{C}) \sim SU(2) \sim Sp(1) \quad D=5,6$

$U(1) \times SU(2) \quad D=4$

$\uparrow$

$SO(2) \times SO(3,1) \subset SO(5,1)$

3 different massless supermultiplets 4+4 dof (D=6)

hypermultiplet	4 scalars + chiral spinor singlet
vector multiplet	vector + chiral spinor doublet
tensor multiplet	dual tensor + scalar + chiral spinor doublet

merge/exchange in 3 spacetime dimensions

$$SO(5,1) \times SU(2) \longrightarrow SO(3,1) \times SU(2) \times SU(2)$$

c-map

$\rightarrow N=4$ susy

Cones

start from

$$\mathcal{L} = \tfrac{1}{2}\, g_{AB}\, \partial_\mu \phi^A \partial^\mu \phi^B - \tfrac{1}{4}\,\frac{d-2}{d-1}\, \chi(\phi)\, R$$

"improvement" term

scale invariance:

$$D_A \partial_B \chi = g_{AB} \qquad \textbf{homothety}$$

$$\rightarrow \quad \chi = \tfrac{1}{2}\, g^{AB} \chi_A \chi_B \quad \text{(locally up to constants)}$$

$$\delta\phi^A = w\, \chi^A \qquad \delta g_{\mu\nu} = -2\, g_{\mu\nu}$$

$\boxed{\text{CONE}}$ $$\chi^A \frac{\partial}{\partial\phi^A} = \frac{\partial}{\partial\phi} \qquad \phi^A \rightarrow \phi, \phi^a$$

$$\chi = e^{2\phi}\, \hat{\chi}(\phi^a)$$

choose

$$\chi = \text{constant} \sim (M_{\text{Planck}})^{d-2}$$

gauge equivalent to Einstein gravity coupled to a nonlinear sigma model with target space $\mathcal{M}$ (Weyl–Stueckelberg)

initial space: Cone over $\mathcal{M}$.

generalize this to supersymmetry

requires CONFORMAL SUPERGRAVITY

(exists up to 16 supercharges)

$\rightarrow$ superconformal quotient

$N=0$ no further structure

$N=1$ χ + phase , Kähler (+ S susy)
Kähler quotient

$N=2$ vector χ + phase , Kähler (+ S susy)
Kähler quotient → special geometry

$N=2$ hyper $\chi + S^3$, hyperkähler (+ S susy)
superconformal quotient
→ quaternion-kähler

Note : compensating chiral, vector, hypermultiplet

χ invariant under R symmetry
(incorporated in conformal s.g)
i.e. $U(1)$ or $SU(2) \sim Sp(1)$

+ "additional" isometries.
(needs qualification)

PM :
Kähler and hyperkähler quotients can be imposed supersymmetrically

Vector multiplets

gauge fields $W_\mu^I \overset{\text{susy}}{\longleftrightarrow}$ scalars X^I

from $D=6$ $\quad W^I_{5,6}(x) \to X^I$ $\begin{cases} \text{real for } D=5 \\ \text{complex for } D=4 \end{cases}$

superconformal quotient

Kähler cone $\longrightarrow$ special Kähler space

$$\chi(X,\bar X) = i\left(F_I(X)\,\bar X^I - X^I\,\bar F_I(\bar X)\right)$$

(note: EM invariant)

projective geometry

X^I sections of a line bundle

section follows from $\delta_S\,\Omega^I{}_\alpha$

on the cone:

complex structure $J^2=-1$

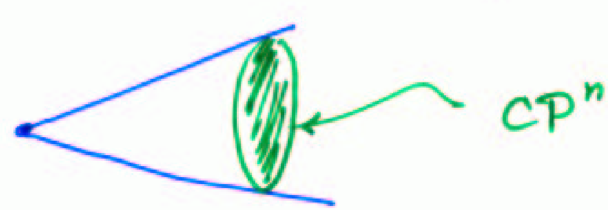
covariantly constant

$U(1)$ isometry $\quad k^A_{U(1)} = J^A{}_B\,\chi^B$

homothety.

$$D_A\,k_B^{U(1)} = -J_{AB}$$

example: Kähler cone ~ flat space



(in supergravity: noncompact versions)

Hypermultiplets

no preferred basis.

$$\mathcal{L} \sim g_{AB}\, \partial_\mu \phi^A \partial^\mu \phi^B \quad + \quad \mathcal{R}\chi(\phi)$$

$\hookrightarrow$ hyperkähler

3 complex structures
holonomy $\subset Sp(n)$
Ricci flat.

$A = 1, \dots 4n$

spinors $\zeta^\alpha \quad \zeta^{\bar\alpha} \qquad \alpha, \bar\alpha = 1, \dots 2n$

$$\mathcal{L} \sim G_{\bar\alpha\beta}\, \bar\zeta^\alpha \not{D} \zeta^\beta$$

supersymmetry
$$\delta\phi^A \propto \gamma^A_{i\bar\alpha}\, \bar\epsilon^i \zeta^{\bar\alpha} + \text{h.c}$$
$$\delta\zeta^\alpha \propto V^\alpha_{A i}\, \not\partial \phi^A \varepsilon^i$$

quaternionic vielbeine γ, V covariantly constant

connections : target space diffs
fermion frame rotations.

$$\gamma_{Ai\bar\alpha}\, \bar V_B^{j\bar\alpha} = \varepsilon_{ik}\, J^{kj}_{AB} + \frac{1}{2}\, g_{AB}\, \delta_i^{\ j}$$

complex structure $\quad$ hyperkähler metric

ETC

Hyperkähler cone

homothety $D_A \partial_B \chi = g_{AB}$ ← hyperkähler potential

$SU(2)$ isometries $\vec{K}_A = \vec{J}_{AB}\, \chi^B$

triholomorphic isometries $\boxed{\vec{\mu} = -\frac{1}{2} \vec{K}_A\, K^A}$

$\vec{\mu}$ moment map: $\partial_A \vec{\mu} = \vec{J}_{AB}\, k^B$
(general)
$\Delta \vec{\mu} = 0$

$\vec{K}^A \quad \chi^A$ 4 orthogonal vectors of length$^2 = 2\chi$

$\chi^A k_A = 0$

horizonal: $\vec{K}^A R_{ABC}{}^D = \chi^A R_{ABC}{}^D = 0$

associated quaternionic bundle (Swann)

$A_i{}^\alpha(\phi)$ section of $Sp(1) \times Sp(n)$ bundle.

$D_B A_i{}^\alpha = V_{Bi}{}^\alpha$

(pseudoreal)
$\boxed{\text{cf } X^I}$: $\delta_s \xi^\alpha$

$\bar{\Omega}_{\alpha\beta}\, A_i{}^\alpha A_j{}^\beta = \varepsilon_{ij}\, \chi$

$\bar{\Omega}_{\alpha\beta}\, A_i{}^\alpha D_B A_j{}^\beta = \frac{1}{2} \varepsilon_{ij}\, \chi_B + k_{ij\,B}$ etc

supersymmetry:

$$\mathcal{L} \propto -G_{\bar\alpha\beta} \left\{ \frac{1}{2} \partial_\mu A_i{}^\beta\, \partial^\mu A^{i\bar\alpha} + \bar{\xi}^{\bar\alpha} \overleftrightarrow{\not{D}}\, \xi^\beta \right\}$$

$$- W_{\bar\alpha\beta\bar\gamma\delta} \left(\bar{\xi}^{\bar\alpha} \gamma^\mu \xi^\beta \right)\left(\bar{\xi}^{\bar\gamma} \gamma_\mu \xi^\delta \right)$$

$+$ potential (gauging)

$W \propto R_{ABC}{}^D$

ALL FOLLOWS FROM SUSY

More on Cones and Quotients

Cone over χ = constant hypersurface

$$G_{AB} = \frac{1}{\chi}\left(g_{AB} - \frac{1}{2\chi}\chi_A \chi_B\right) \qquad \text{(horizonal)}$$

(3) - Sasakian metric

quotient isometry (left invariant by χ)

$$G_{AB} = \frac{1}{\chi}\left(g_{AB} - \frac{1}{2\chi}\chi_A \chi_B - \frac{1}{k^2} k_A k_B\right)$$

k^3 isometry → k^{3A} holomorphic ; χ moment map

→ superconformal / Kähler quotient

metric is horizontal and independent of the orthogonal coordinates (special geometry / vector multiplets)

For HKC → twistor space (2n-1)-dimensional.

Sp(1) superconformal quotient

$$G_{AB} = \frac{1}{\chi}\left(g_{AB} - \frac{1}{2\chi}\left(\chi_A \chi_B + \vec{k}_A \cdot \vec{k}_B\right)\right)$$

quaternion-Kähler metric ! 4n-4 coordinates

required explicit gauge fixing.

Summarize:

HKC : cone over 3-Sasakian space

Kähler quotient → twistor space

twistor space S^2 fibration of QK^{4n-4}

3 Sasakian space S^3 fibration of QK^{4n-4}

ASSERTION HKC and/or twistor space are the best way to deal with hypermultiplets

e.g. quaternion-Kähler spaces

Results (with Vandoren, Kleyn, Roček, ...)

- general Lagrangian (with gauging) in terms of Swann sections, both for HKC and QK space, prior to the last (quaternionic/twistor) gauge condition

  question formulate this explicitly in twistor space ?

- explicit construction

  HKC $\leftrightarrow$ twistor space $\leftrightarrow$ QK space
  geometry including complex/quaternionic structure
  triholomorphic $\leftrightarrow$ quaternionic isometries

- Legendre transform
  superconformal tensor multiplets $\leftrightarrow$ HKC
  determine the maximal number of abelian isometries
  (does not exclude nonabelian ones)

- gauging: potential both on the HKC and the QK space formulation.

  crucial $$\mu_{\text{CONE}} = \mu_{\substack{SK \\ QK}}\, \chi$$

  up to a phase / Sp(1) rotation

- quaternion-K gauge choice
  some of $\vec{\mu}_{QK}$ vanish
  QK quotients seem to become singular
  $\rightarrow$ change order hyperkähler quotient prior to s.c. quotient
  [relevant for Higgs branch dynamics below vector boson mass scale]

Application (work in progress)

Abelian gauging of quaternion-kähler spaces (dim $4n$) whose target space metric has $n+1$ abelian isometries (possibly complemented to nonabelian motions)

This covers all symplectic and unitary Wolf spaces

Two functions:

$F(X)$ governs the vector multiplets

$H(x, v, \bar{v})$ governs the hyper multiplets $(v, \bar{v}, \underbrace{w, \bar{w}})^P$

related to $v, \bar{v}, x$ by Legendre transform

Abelian isometries $w_P \to w_P + i\Lambda_P$

Each isometry has coupling constant g_P

$$\mathcal{L}_{scalar} = - g_P g_Q \left[-4 H^{PQ} X^P \bar{X}^Q + N^{PQ} (x^P x^Q + 4 v^P \bar{v}^Q) \right]$$

+ nonabelian gauging for vectors only

$$H_{PQ} = \frac{\partial^2 H}{\partial x^P \partial x^Q} \qquad N_{IJ} = -i(F_{IJ} - \bar{F}_{IJ})$$

part of HKC metric — Kähler cone metric

Observe H^{PQ} N^{PQ} not necessarily positive!

$\mathcal{L}_{scalar}$ defined on the HK cone $\times$ K cone.

$$\chi = 2 X^I N_{IJ} \bar{X}^J$$

Minimal case

$F(X)$ quadratic polynomial signature $(- + + + \cdots)$
$\rightarrow$ vectors parametrize CP_m

$H(x, v, \bar{v})$

$$\chi = \sum_P \sigma_P \sqrt{(x^P)^2 + 4\, v^P \bar{v}^P}$$

$- + + + +$

$\rightarrow$ hypers parametrize HP_n

(class discussed by dW, Lauwers, Philippe, Su, VanProeyen PLB 134 (1984))

$$\mathcal{L} = -4\, g_P^2\, \sigma_P \sqrt{(x^P)^2 + 4\, v^P \bar{v}^P}\, |X^P|^2$$

$$- g_P g_Q\, N^{PQ} \left(x^P x^Q + 4\, v^P \bar{v}^Q \right)$$

constant.

When taking account of the cone structure the potential is invariant under the $SU(2) \times U(1)$ R-symmetry. This degenerag is not physical and can be disregarded.