# Type IIB Supergravity, D3 branes and ALE manifolds

*Based on recent work by:*

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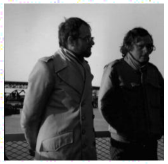
# Introduction



- There are interesting solutions of type IIB theory, named fractional D3 branes.
  - **The gauge duals are non-conformal N=2 gauge theories in d=4**
- Fractional branes are commonly viewed as 5-branes wrapped on a vanishing cycle of transverse space
  - **Transverse space is $R^2$ x $R^4/\Gamma$.**
- We have found a supersymmetric (BPS) D3-brane solution where transverse space is $R^2$ x ALE
  - **In the orbifold limit we recover fractional branes**
  - **The warp factor is determined by a harmonic equation on ALE**
  - **In Eguchi Hanson case the harmonic equation reduces to a** *confluent Heun equation*
  - **Open questions on the boundary action and the gauge dual.**

# Type IIB Sugra



**FIELD CONTENT OF TYPE IIB SUPERGRAVITY.**

Greek indices $\alpha, \beta, \ldots = 1, 2$ run in the fundamental representation of SU(1,1), $C^{[p]}$ denote the Ramond-Ramond $p$–forms: RED=fermion, Blue =boson

| Fields | SU(1,1) | U(1) | superstring zero modes |
|---|---|---|---|
| $V^{\underline{a}}_{\underline{\mu}}$ | $J=0$ | 0 | graviton $g_{\underline{\mu\nu}}$ |
| $\psi_{\underline{\mu}}$ | J=0 | $\frac{1}{2}$ | gravitinos $\psi_{A\underline{\mu}}$ |
| $A^{\alpha}_{\underline{\mu\nu}}$ | $J=\frac{1}{2}$ | 0 | $B_{[2]}$ , $C_{[2]}$ |
| $C_{\underline{\mu\nu\rho\sigma}}$ | $J=0$ | 0 | $C_{[4]}$ |
| $\lambda$ | $J=0$ | $\frac{3}{2}$ | dilatinos $\lambda_A$ |
| $\mathbb{L}^{\alpha}{}_{\beta}$ | $J=\frac{1}{2}$ | $\pm 1$ | $\varphi, C_{[0]}$ |

*The D3 brane couples to $C_{[4]}$ whose field strength involves a Chern Simons of lower forms*

The bosonic field strengths are defined as follows:

$$F^{RR}_{[1]} = dC_{[0]} \quad ; \quad F^{RR}_{[3]} = dC_{[2]} - C_{[0]}\, dB_{[2]}$$

$$F^{NS}_{[3]} = dB_{[2]} \quad ; \quad F^{RR}_{[5]} = dC_{[4]} - \frac{1}{2}\left(B_{[2]} \wedge dC_{[2]} - C_{[2]} \wedge dB_{[2]}\right) \quad ; \quad F^{RR}_{[5]} = \star F^{RR}_{[5]}$$

*Castellani & Pesando (1991) established geometric formulation*

# The SL(2,R)~SU(1,1) structure



SL(2,$\mathbb{R}$) *Lie algebra*

$$[L_0\,,\,L_\pm] = \pm\, L_\pm \quad ; \quad [L_+\,,\,L_-] = 2\,L_0$$

*The solvable Lie algebra parametrization of the coset naturally introduces the dilaton and the RR 0-form*

$$\mathbb{L}\,(\varphi, C_{[0]}) = \exp\left[\varphi\, L_0\right] \exp\left[C_{[0]} e^{\varphi}\right] = \begin{pmatrix} \exp[\varphi/2] & 0 \\ C_{[0]} & \exp[-\varphi/2] \end{pmatrix}$$

where $\varphi(x)$ and $C_{[0]}$ are the dilaton and the Ramond-Ramond 0-form.

$$\Lambda^{-1}\,d\Lambda = \begin{pmatrix} -\mathrm{i}Q & P \\ P^\star & \mathrm{i}Q \end{pmatrix} \quad ; \quad \mathcal{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix} \quad ; \quad \mathrm{SU}(1,1) \ni \Lambda = \mathcal{C}\,\mathbb{L}\,\mathcal{C}^{-1}$$

Explicitly

$$P = \tfrac{1}{2}\left(d\varphi - \mathrm{i}\, e^{\varphi}\, dC_{[0]}\right) \quad \text{scalar vielbein}$$

$$Q = \tfrac{1}{2}\exp[\varphi]\, dC_{[0]} \quad \mathrm{U}(1)\text{-connection}$$

# The FDA defining the Theory

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*The curvatures of the free differential algebra in the complex basis*

***Note the Chern Simons***

$$
\begin{aligned}
R^a &= \mathcal{D}V^a - i\bar{\psi}\wedge\Gamma^a\psi \\
R^{ab} &= d\omega^{ab} - \omega^{ac}\wedge\omega^{db}\,\eta_{cd} \\
\rho &= \mathcal{D}\psi \equiv d\psi - \tfrac{1}{4}\omega^{ab}\wedge\Gamma_{ab}\psi - \tfrac{1}{2}\mathrm{i}Q\wedge\psi \\
\mathcal{H}^{\alpha}_{[3]} &= dA^{\alpha}_{[2]} + 2i\Lambda^{\alpha}_{+}\bar{\psi}\wedge\Gamma_a\psi^{\star}\wedge V^a + 2\mathrm{i}\Lambda^{\alpha}_{-}\bar{\psi}^{\star}\wedge\Gamma_a\psi\wedge V^a \\
\mathcal{F}_{[5]} &= dC_{[4]} + \tfrac{1}{16}\,\mathrm{i}\,\epsilon_{\alpha\beta}A^{\alpha}_{[2]}\wedge\mathcal{H}^{\beta}_{[3]} + \mathrm{i}\tfrac{1}{6}\,\bar{\psi}\wedge\Gamma_{abc}\psi\wedge V^a\wedge V^b\wedge V^c \\
&\quad + \mathrm{i}\tfrac{1}{4}\,\epsilon_{\alpha\beta}A^{\alpha}_{[2]}\wedge\left(\Lambda^{\beta}_{+}\bar{\psi}\Gamma_a\psi^{\star} + \Lambda^{\beta}_{-}\bar{\psi}^{\star}\Gamma_a\psi\right)\wedge V^a \\
\mathcal{D}\lambda &= d\lambda - \frac{1}{4}\omega^{ab}\Gamma_{ab}\lambda - \mathrm{i}\tfrac{3}{2}\,Q\lambda \\
\mathcal{D}\Lambda^{\alpha}_{\pm} &= d\Lambda^{\alpha}_{\pm} \mp \mathrm{i}\,Q\,\Lambda^{\alpha}_{\pm}.
\end{aligned}
$$

$$\mathbb{C}\bar{\psi}^T = \psi^{*}$$

*Castellani & Pesando (1990-1991)*

***As for all supergravities the algebraic structure is encoded in an FDA. (D'Auria, Fré, (1982), Castellani, P. Van Nieuwenhuizen, K. Pilch (1982)***

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# Supergravity People @ Stony Brook 1982

# Always running to get ahead .........


**while Sergio relaxes in L.A!**

***Ops!.. I was also there but do not tell anybody....***

# The rheonomic parametrizations~susy rules

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$$
\begin{aligned}
R^a &= 0 \\
\rho &= \rho_{ab} V^a \wedge V^b + \frac{5}{16} \mathrm{i} \Gamma^{a_1 - a_4} \psi V^{a_5} \left( F_{a_1 - a_5} + \frac{1}{5!} \epsilon_{a_1 - a_{10}} F_{a_6 - a_{10}} \right) \\
&\quad + \frac{1}{32} \left( -\Gamma^{a_1 - a_4} \psi^* V_{a_1} + 9 \Gamma^{a_2 a_3} \psi^* V^{a_4} \right) \Lambda^\alpha_+ \mathcal{H}^\beta_{a_2 - a_4} \epsilon_{\alpha\beta} + \text{fermion bilinears} \\
\mathcal{H}^\alpha_{[3]} &= \mathcal{H}^\alpha_{abc} V^a \wedge V^b \wedge V^c + \Lambda^\alpha_+ \bar{\psi}^* \Gamma_{ab} \lambda^* V^a \wedge V^b + \Lambda^\alpha_- \bar{\psi} \Gamma_{ab} \lambda V^a \wedge V^b \\
\mathcal{F}_{[5]} &= F_{a_1 - a_5} V^{a_1} \wedge \ldots \wedge V^{a_5} \\
\mathcal{D}\lambda &= \mathcal{D}_a \lambda V^a + \mathrm{i} P_a \Gamma^a \psi^* - \frac{1}{8} \mathrm{i} \Gamma^{a_1 - a_3} \psi \epsilon_{\alpha\beta} \Lambda^\alpha_+ \mathcal{H}^\beta_{a_1 - a_3} \\
\mathcal{D}\Lambda^\alpha_+ &= \Lambda^\alpha_- P_a V^a + \Lambda^\alpha_- \bar{\psi}^* \lambda \\
R^{ab} &= R^{ab}{}_{cd} V^c \wedge V^d + \text{fermionic terms}
\end{aligned}
$$

***Fermionic Components are expressed in terms of space-time components***

# Our task: the equations to be solved

The Bosonic field equations

$$
\begin{aligned}
R^{pr}{}_{qr} - \tfrac{1}{2}\delta^p_q R^{ab}{}_{ab} \;=\; & -75\left(F_{qa_1-a_4}F^{pa_1-a_4} - \frac{1}{10}\delta^p_q F_{a_1-a_5}F^{a_1-a_5}\right) \\
& -\frac{9}{16}\left(\widehat{\mathcal{H}}^{pa_1a_2}_{+}\widehat{\mathcal{H}}_{-|qa_1a_2} + \widehat{\mathcal{H}}^{pa_1a_2}_{-}\widehat{\mathcal{H}}_{+|qa_1a_2} - \frac{1}{3}\delta^p_q \widehat{\mathcal{H}}^{a_1a_2a_3}_{+}\widehat{\mathcal{H}}_{-|a_1a_2a_3}\right) \\
& -\tfrac{1}{2}\left(P^p P^*_q + P_q P^{*p} - \delta^p_q P^a P^*_a\right)
\end{aligned}
$$

$$\mathcal{D}^a P_a \;=\; -\tfrac{3}{8}\,\widehat{\mathcal{H}}^{a_1a_2a_3}_{+}\widehat{\mathcal{H}}_{+|a_1a_2a_3}$$

$$\mathcal{D}^b\widehat{\mathcal{H}}_{+|a_1a_2b} \;=\; -\mathrm{i}20\,{}_{a_1a_2b_1b_2b_3}\widehat{\mathcal{H}}^{b_1b_2b_3}_{+} + P^b\widehat{\mathcal{H}}_{-|a_1a_2b}$$

$$\mathcal{D}^b F_{a_1a_2a_3a_4b} \;=\; \mathrm{i}\frac{1}{160}\epsilon_{a_1a_2a_3a_4b_1\dots b_6}\widehat{\mathcal{H}}^{b_1b_2b_3}_{+}\widehat{\mathcal{H}}^{b_4b_5b_6}_{-}$$

- where dressed field strengths are:

$$\widehat{\mathcal{H}}_{\pm|a_1a_2a_3} \;=\; \epsilon_{\alpha\beta}\Lambda^\alpha_\pm\,\mathcal{H}^\beta_{a_1a_2a_3}$$

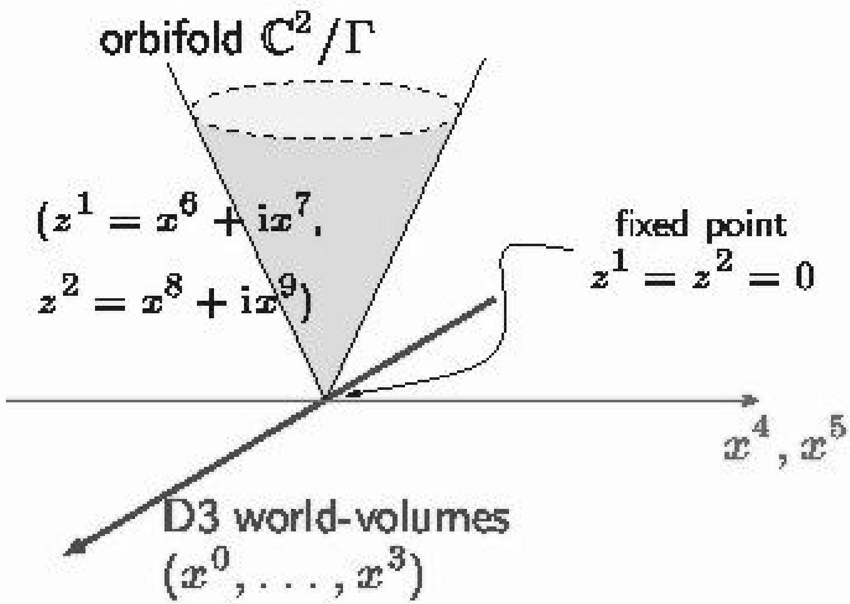
Note that these equations are written in flat indices and appear as cosntraints on the curvature components

# Fractional Branes= D3 branes on orbifolds $C^2/\Gamma$

- Consider Type IIB string theory on a space-time of the form

$$\mathbb{R}^{1,3} \times \mathbb{R}^2 \times \mathbb{C}^2/\Gamma \; ,$$

in which (fractional) D3-branes, transverse to the orbifold, are located at the orbifold singularity:




***Susy is halved in the bulk by restricted holonomy***

In the dual gauge theory, **$x^4+ix^5$** makes the complex scalar of the vector multiplet while

**$x^6, x^7, x^8, x^9$** constitute the scalar part of a hypermultiplet

# ALE manifolds as orbifold resolutions



- The group $\Gamma$ is a discrete subgroup of $\mathrm{SU}(2)$ acting on $\mathbb{C}^2$ by

$$g \in \Gamma : \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \mapsto \mathcal{Q}(g) \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} ,$$

  with $\mathcal{Q}(g) \in \mathrm{SU}(2)$ the defining 2-dim representation.

- Such Kleinian subgroups $\Gamma$ are ADE-classified. For instance,

$$\mathrm{A}_{n-1} \leftrightarrow \Gamma = \mathbb{Z}_n \; : \; \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \sim \begin{pmatrix} e^{2\pi i/n} \, z^1 \\ e^{-2\pi i/n} \, z^2 \end{pmatrix} ;$$

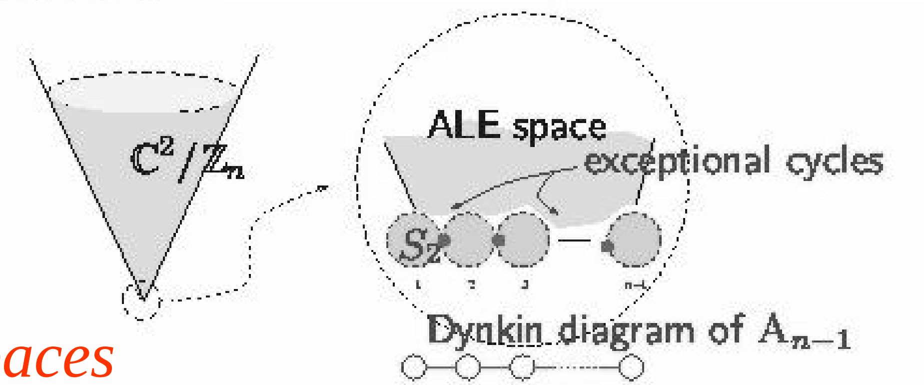
  Also, $D_{n+2} \leftrightarrow \Gamma = \mathbb{D}_n$, and $E_{6,7,8} \leftrightarrow \Gamma = \mathcal{T}, \mathcal{O}, \mathcal{I}$.

- Resolving the singularity produces an ALE space,with non-trivial two-cycles $e_i$ whose intersection is

$$\boxed{e_i \cdot e_j = -C_{ij} \; ,}$$

  where $C_{ij}$ is the Cartan matrix associated to $\Gamma$.

***ALE manifolds are related to the ADE classification of Lie algebras. They can be obtained as suitable Hyperkahler quotients of flat HyperKahler manifolds***




*Kronheimer construction of ALE spaces is realized by String Theory*

# ALE Manifolds: main relevant feature

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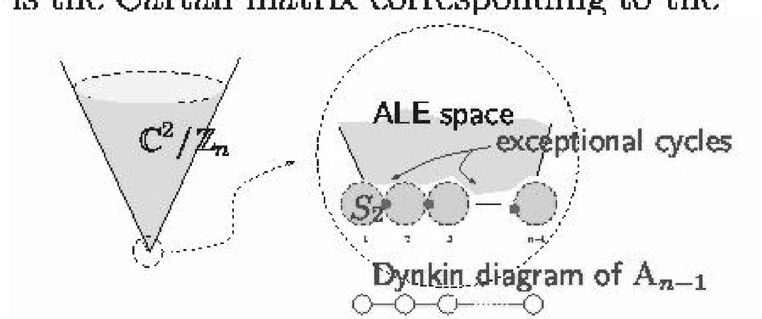
Generalities on ALE spaces:

Self dual solutions of 4 dim. Euclidean Einstein gravity whose asymptotic geometry is $\mathbb{R}^4/\Gamma$ $(\Gamma \subset SU(2))$.
They have compact cycles $\Sigma_I$, $I = 1, \dots, \tau - 1$ ($\tau$ being the Hirzebruch signature) and correspondingly $\tau - 1$ square normalizable *anti–selfdual* harmonic forms $\omega^I$:

$$\int_{\Sigma_K} \omega^J = \delta^I_K \; ; \; \int_{ALE} \omega^I \wedge \omega^J = -\mathcal{C}^{-1\,IJ}$$

$$\omega^I \wedge \omega^J = -\Delta^{IJ}(y)\,\Omega_{ALE}$$

where $y^\sigma$ denote local coordinates on ALE, $\mathcal{C}_{IJ}$ is the Cartan matrix corresponding to the ADE Dynkin diagram of $\Gamma$ and $\Delta^{IJ}(y) > 0$

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# The ansatz:

**Ansatz for the 3–brane solution on an ALE space with non trivial $\mathcal{H}_+$ flux:**

- **Metric:** *warped* $\mathbb{R}^{1,3} \times \mathbb{R}^2 \times ALE$ (coordinates $x^\mu$, $z$, $\bar{z}$ (or $x^4$, $x^5$) and $y^\sigma$):

$$ds^2 = H^{-\frac{1}{2}}(-\eta_{\mu\nu}dx^\mu\, dx^\nu) + H^{\frac{1}{2}}dzd\bar{z} + H^{\frac{1}{2}}ds^2_{ALE}$$

  where $H = H(z, \bar{z}, y)$.

- **Axion/Dilaton:** $C_{[0]} = \phi = 0$.
- **Complex 3–Form:** $\mathcal{H}_+ = 2\, d\gamma_I(z,\bar{z}) \wedge \omega^I$; $\mathcal{H}_- = -2\, d\overline{\gamma}_I(z,\bar{z}) \wedge \omega^I$
- **5–Form:** $F^{R-R}_{[5]} = U + \star U$ where $U = d\left(H^{-1}\,\Omega_{\mathbb{R}^{1,3}}\right)$

Note the essential use of the harmonic 2 forms dual to the homology cycles of ALE

**Axion/Dilaton equations:**

$$\mathcal{H}_+ \wedge \star\mathcal{H}_+ = 0 \Leftrightarrow [\mathcal{H}_+ \text{self (anti self) dual} \Leftrightarrow \gamma_I = \gamma_I(z) \;\; (\overline{\gamma}_I = \overline{\gamma}_I(\bar{z}))]$$

$$\star_6\mathcal{H}_\pm = \pm \mathrm{i}\,\mathcal{H}_\pm \Leftrightarrow \boxed{\gamma_I = \gamma_I(z)}$$

→ Holomorphic field

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# Pinpointing the sources:

**$C_{[2]}$ and $B_{[2]}$ equations:**

$$d \star \mathcal{H}_+ = \mathrm{i}\, F^{RR}_{[5]} \wedge \mathcal{H}_+ + \mathcal{T}(z,\bar{z},y) \;\Leftrightarrow\; \Box_{\mathbb{R}^2}\, \gamma_I = \mathrm{T}(z,\bar{z})$$

where $\mathcal{T}(z,\bar{z},y)$ depends on the interaction of the 3–brane with $C_{[2]}$ and $B_{[2]}$ $\Rightarrow$ derived from boundary action (unknown at this stage), it fixes boundary conditions on $\gamma_I(z)$:

$$\gamma_I(z) \;\propto\; z^k \;\; (k \geq 1)$$

$$\gamma_I(z) \;\propto\; \log(z) \;\Leftrightarrow\; \begin{cases} \mathrm{T}(z,\bar{z}) \propto \delta^2(z,\bar{z}) & \text{i.e. 3–brane source of } C_{[2]} \text{ and } B_{[2]} \\ \textit{localized} \text{ in } \mathbb{R}^2, & \text{as for fractional branes on an orbifold.} \end{cases}$$

This is the real problem of interpretation: what is the source of B,C fields?

**$C_{[4]}$ Equation + Self–Duality of $F^{RR}_{[5]}$:**

$$d\, F^{RR}_{[5]} = \mathrm{i}\,\tfrac{1}{8}\,\mathcal{H}_+ \wedge \mathcal{H}_- + \mathcal{S}(z,\bar{z},y) \;\Leftrightarrow$$

$$\left(\Box_{\mathbb{R}^2} + \Box_{ALE}\right) H(z,\bar{z},y) = -\partial_z \gamma_I\, \partial_{\bar{z}} \bar{\gamma}_J\, \Delta^{IJ}(y) + \mathrm{S}(z,\bar{z},y)$$

where $\mathcal{S}(z,\bar{z},y)$ is the D3–brane charge source term, to be considered in the present *macroscopic* analysis as a choice of boundary conditions.

**Even without D3 brane charge there is effective source for 5-form**

# Killing spinors I:

**Counting SUSYs (in brief)**

Looking for 10 dimensional Killing spinor $\chi(z, \bar{z}, y)$ $(\Gamma^{11}\chi = \chi)$: $\delta_\chi \psi_M = \delta_\chi \lambda = 0$.

- Case $\mathcal{H}_+ = 0$ :
  - Presence of D3 brane $(F^{RR}_{[5]} \neq 0)$ $\Rightarrow$ $\boxed{\mathbf{32}} \rightarrow \boxed{\mathbf{16}}$
  - Curvature of ALE space $(SU(2)$ holonomy$)$ $\Rightarrow$ $\boxed{\mathbf{16}} \rightarrow \boxed{\mathbf{8}}$
- Switching $\mathcal{H}_+$ on, above solution $\chi$ is still a Killing spinor iff:

$$\boxed{\star_6 \mathcal{H}_+ = \mathrm{i}\,\mathcal{H}_+}$$

consistently with our ansatz.

Theory on 3 brane world volume is an $\mathcal{N} = 2$, $D = 4$ gauge theory.

# Killing spinors II:

**Counting SUSYs**

Looking for 10–dimensional Killing spinor $\chi(z, \bar{z}, y)$ $(\Gamma^{11}\chi = \chi)$: $\delta_\chi \psi_M = \delta_\chi \lambda = 0$.

$$\delta\psi_M = \mathcal{D}_M\chi + \frac{1}{16}\mathrm{i}\Gamma^{A_1\dots A_5}F_{A_1\dots A_5}V^B_M\Gamma_B\chi + \frac{1}{32}\left(-\Gamma_{BA_1\dots A_3}V^B_M + 9\Gamma^{A_1A_2}V^{A_3}_M\right)\mathcal{H}_{+|A_1A_2A_3}\chi^\star$$

$$\delta\lambda = -\mathrm{i}\frac{1}{8}\Gamma^{A_1A_2A_3}\mathcal{H}_{+|A_1A_2A_3}\chi$$

On geometry $\mathbb{R}^{1,3} \times (\mathbb{R}^2 \times ALE)$; $D = 10 = 4 + (6 = 2 + 4) \to \chi = \epsilon \otimes (\eta = \theta \otimes \xi)$

The Clifford algebra splits:

$\Gamma^A = \{\gamma^a \otimes \mathbb{1}; \gamma^5 \otimes \tau^i\}$ ; $\tau^i = \{\sigma^\alpha \otimes \mathbb{1}; \mathrm{i}\sigma^3 \otimes \bar{\gamma}^u\} \to (\gamma^5\epsilon) \otimes (\tau_\bullet\eta) = \mathrm{i}\,\epsilon \otimes \eta$ $\boxed{32}$.

Counting solutions with $\mathcal{H}_+ = 0$ :

a) $D3$–brane $(F^{RR}_{[5]} \neq 0) \Rightarrow (\gamma^5\epsilon) = \epsilon$; $(\tau_\bullet\eta) = \mathrm{i}\,\eta$ $\boxed{16}$

b) $[\mathbb{R}^2 \times ALE$ geometry$] + [\chi = H^{-1/8}\epsilon \otimes \widehat{\eta}] \Rightarrow$

$$[\widehat{\mathcal{D}}_{ALE}\widehat{\eta} = 0 \Rightarrow R^{uv}\bar{\gamma}_{uv}\xi = \frac{1}{2}R^{uv}\bar{\gamma}_{uv}(1-\bar{\gamma}^5)\xi = 0 \Leftrightarrow \bar{\gamma}^5\xi = \xi] \quad \boxed{8}$$

(self–duality of ALE curvature $R^{uv}$ )

# Killing spinors III:

In the case $\mathcal{H}_+ \neq 0$, previous solution defining **8** residual s. charges, is still a Killing spinor iff:

$$\begin{bmatrix} \left(\Gamma^{A_1A_2A_3}\,\mathcal{H}_{+|A_1A_2A_3}\right)\,\chi \\ \\ \left(\Gamma^{A_1A_2A_3}\,\mathcal{H}_{+|A_1A_2A_3}\right)\,\chi^\star \end{bmatrix} \Leftrightarrow \begin{bmatrix} \text{(a)+(b)} \\ + \\ \star_6\mathcal{H}_+ \;=\; \mathrm{i}\,\mathcal{H}_+ \end{bmatrix}$$

consistently with our ansatz.

Summarizing the conditions on the Killing spinor:

- $\chi = H^{-1/8}\,\epsilon \otimes \widehat{\eta}\,;\; \gamma^5\,\epsilon = \epsilon\,;\; \tau_\bullet\widehat{\eta} = \mathrm{i}\,\widehat{\eta}$
- $\widehat{\eta} = \theta \otimes \widehat{\xi}\,;\; \bar{\gamma}^5\,\widehat{\xi} = \widehat{\xi}$
- $\widehat{\mathcal{D}}_{ALE}\,\widehat{\eta} = 0 \;\Rightarrow\;$ 2 solutions , **8** s. charges

Two solutions because of SU(2) holonomy of ALE

# Harmonic equation in the

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## Equation for H in the **Eguchi–Hanson Case**

Simplest case: ALE asymptotically $\mathbb{R}^4/\mathbb{Z}_2$, $\tau = 2 \Rightarrow 1$ exceptional cycle $\Sigma \leftrightarrow$ anti self dual harmonic 2 form $\omega$

Eguchi Hanson metric:

$$ds^2_{EH} = g(r)^{-2}\,dr^2 + \frac{r^2}{4}\left(d\theta^2 + \sin(\theta)^2\,d\phi^2\right) + \frac{r^2}{4}\,g(r)^2\,\left(d\psi + \cos(\theta)\,d\phi\right)^2$$

$$g(r)^2 = 1 - \left(\frac{a}{r}\right)^4 \;;\quad \{\theta : 0 \to \pi\};\quad \{\psi,\phi : 0 \to 2\pi\};\quad \{r : a \to \infty\ (a > 0)\}$$

$$\omega = \frac{a^2}{4\pi}\,d\left[\frac{d\psi + \cos(\theta)\,d\phi}{r^2}\right] \Rightarrow \omega\wedge\omega = -\frac{2\,a^4}{\pi^2\,r^8}\,\Omega_{ALE}$$

**Explicit form of the compact harmonic 2-form and "intersection function"**

- $\Sigma$ is an $S_2$ located at $r = a$ and spanned by $\{\theta,\phi\}$
- near–cycle behavior: $v = \sqrt{\frac{r^4 - a^4}{4r^2}}$, $v \approx 0$:

$$ds^2_{EH} = \underbrace{dv^2 + v^2\,d\psi^2}_{\mathbb{R}^2} + \underbrace{\frac{a^2}{4}\left(d\theta^2 + \sin(\theta)^2\,d\phi^2\right)}_{S_2} + O(v^4) \qquad \mathbb{R}^2 \times S_2$$

**Note factorization of near cycle metric**



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# The partial Fourier transform

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Equation for $H(z,\bar{z},y)$ in the Eguchi Hanson metric:

$$\left(\partial_4^2+\partial_5^2\right)\,H+\frac{1}{r^3}\partial_r\left(r^3\,g(r)^2\,\partial_r H\right)\;=\;-\frac{2\,a^4}{\pi^2\,r^8}\left|\partial_z\,\gamma\right|^2+S(z,\,\bar{z},\,y)$$

**Boundary conditions**

$$\left(\rho\;=\;|z|^2\to\infty\;;\;r\to\infty\right)\qquad H(z,\,\bar{z},y)\to 1$$

(near–cycle) $\left\{\begin{array}{c}\text{D3–brane charge } Q \textit{ localized} \text{ at}\\ \{O\}\times\Sigma\subset\mathbb{R}^2\times ALE\\ \Downarrow\\ S(z,\,\bar{z},\,y)=-\frac{Q}{2\pi r^3}\,\delta^2(z,\,\bar{z})\,\delta(r-a)\end{array}\right.$

Makes Laplace Equation on ALE x $R^2$ inhomogeneous

Looking for $H(z,\,\bar{z},y)\;=\;H(\rho,\,r)$.
A convenient approach is to Fourier transform the equation along $\mathbb{R}^2$:

$$f(z,\,\bar{z})\;\to\;\mathcal{F}[f](p,\,\bar{p}),\qquad (p,\,\bar{p}\text{ being the conjugate variables to } z,\,\bar{z}).$$

Denoting by $\mu^2=|p|^2$ we define:

$$\begin{aligned}\widetilde{H}(\mu,\,r)\;&\equiv\;\mathcal{F}[H-1](\mu,\,r)\\ j(\mu)\;&=\;\mathcal{F}[\,|\partial\gamma|^2\,](\mu)\end{aligned}$$

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# Heun confluent equation

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The Equation in the new variables has the form:

**D3 brane charge**

$$\frac{1}{r^3}\,\partial_r\left(r^3\,g(r)^2\,\partial_r\widetilde{H}\right)-\mu^2\,\widetilde{H} \;=\; -\frac{2\,a^4}{\pi^2\,r^8}\,j(\mu)-\frac{Q}{2\,\pi\,r^3}\,\delta(r-a)$$

- The above equation is a confluent form of a non homogeneous Heun's equation: its homogeneous part in the complex $r$ plane has two regular singularities $r=a,\,0$ and one irregular singularity at $r=\infty$ (type $[0,2,1_2]$...Hard to handle!)
- General solution has the form:

**Fixed by boundary conditions**

$$\widetilde{H}(\mu,\,r) \;=\; \beta_1\,\widetilde{H}_1(\mu,\,r)+\beta_2\,\widetilde{H}_2(\mu,\,r)+\widetilde{H}^{n-h}(\mu,\,r)$$

$\widetilde{H}_{1,2}$ being two independent solutions of the homogeneous part and $\widetilde{H}^{n-h}$ a particular solution of the whole equation.

**Preliminary Analysis:**

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- Determine $\beta_{1,2}$ from boundary conditions by solving equation in the limits $r\to\infty$ and $r\to a$ (near cycle). Consider first the homogeneous part ($j(\mu)=0\Leftrightarrow\gamma=\text{const.}$) with the boundary condition represented by the source term $S(\mu,\,r)$...

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# Asymptotics and the charges

The asymptotic behaviors of the solution are:

| $r \sim a\ (\mu \to \infty)$ | $r \to \infty$ |
|---|---|
| $\widetilde{H}_1 \sim \frac{1}{a^2} K_0(\mu\, v)$ | $\widetilde{H}_1 \sim \frac{\mu}{r} K_1(\mu\, r)$ |
| $\widetilde{H}_2 \sim 1$ | $\widetilde{H}_2 \sim \frac{2}{\mu\, r} I_1(\mu\, r)$ |

Implementing the boundary conditions:

$$\text{Source term } S(\mu, r) = \frac{Q}{2\,\pi\, r^3}\delta(r-a) \Rightarrow \beta_1 = \frac{Q}{4\,\pi}$$

$$\widetilde{H}(\mu, r \to \infty) = 0 \Rightarrow \beta_2 = 0$$

**The D3 brane charge determines the coefficient of the irregular solution near the cycle**

Asymptotic flatness fixes the coefficient of the irregular solution at *infinity*

$\Downarrow$ (back to the $z$, $\bar{z}$ coordinates)

$$(H(\rho, r) - 1)_{|\gamma = const.} = \begin{cases} \frac{Q}{(x_4^2 + x_5^2 + \ldots + x_9^2)^2} & (\text{for } \rho,\ r \to \infty) \quad \left[\begin{array}{c}\text{Harmonic Potential on}\\ \mathbb{R}^2 \times \mathbb{R}^4/\mathbb{Z}_2\end{array}\right] \\ \frac{Q/2a^2}{(\rho^2+v^2)} & (\text{close to ALE cycle}) \quad \left[\begin{array}{c}\text{Harmonic Potential on}\\ \mathbb{R}^2 \times \mathbb{R}^2 \times S_2\end{array}\right] \end{cases}$$

solution consistent with picture of $D3$–brane "smeared" on the cycle $\Sigma$ with charge density $Q/(\pi\, a^2)$

P. Fré

# Power series and questions

### Study of the solution in $H$

- The Fourier transformed solution:

$$\widetilde{H}(\mu, r) = \frac{Q}{4\pi}\widetilde{H}_1 + \widetilde{H}^{n-h}$$

was then determined in a *finite* neighborhood of $r = a$ in terms of a power series expansion:

$$\widetilde{H}(\mu, r) = \sum_{n=0}^{\infty} c_n(\mu;\gamma)\, v^n$$

and the generic coefficient $c_n(\mu;\gamma)$ was computed as a functional of $\gamma(z) \Rightarrow$ corrections to the previously described asymptotic behavior could be computed.

- In the case $\gamma(z) = K \log(z/\rho_e)$ we may summarize the asymptotic behaviors of $H$:

$$(\rho, r \gg a) : \quad H(\rho, r) \sim \frac{Q}{(\rho^2 + r^2)^2}(1 + \log \text{terms})$$

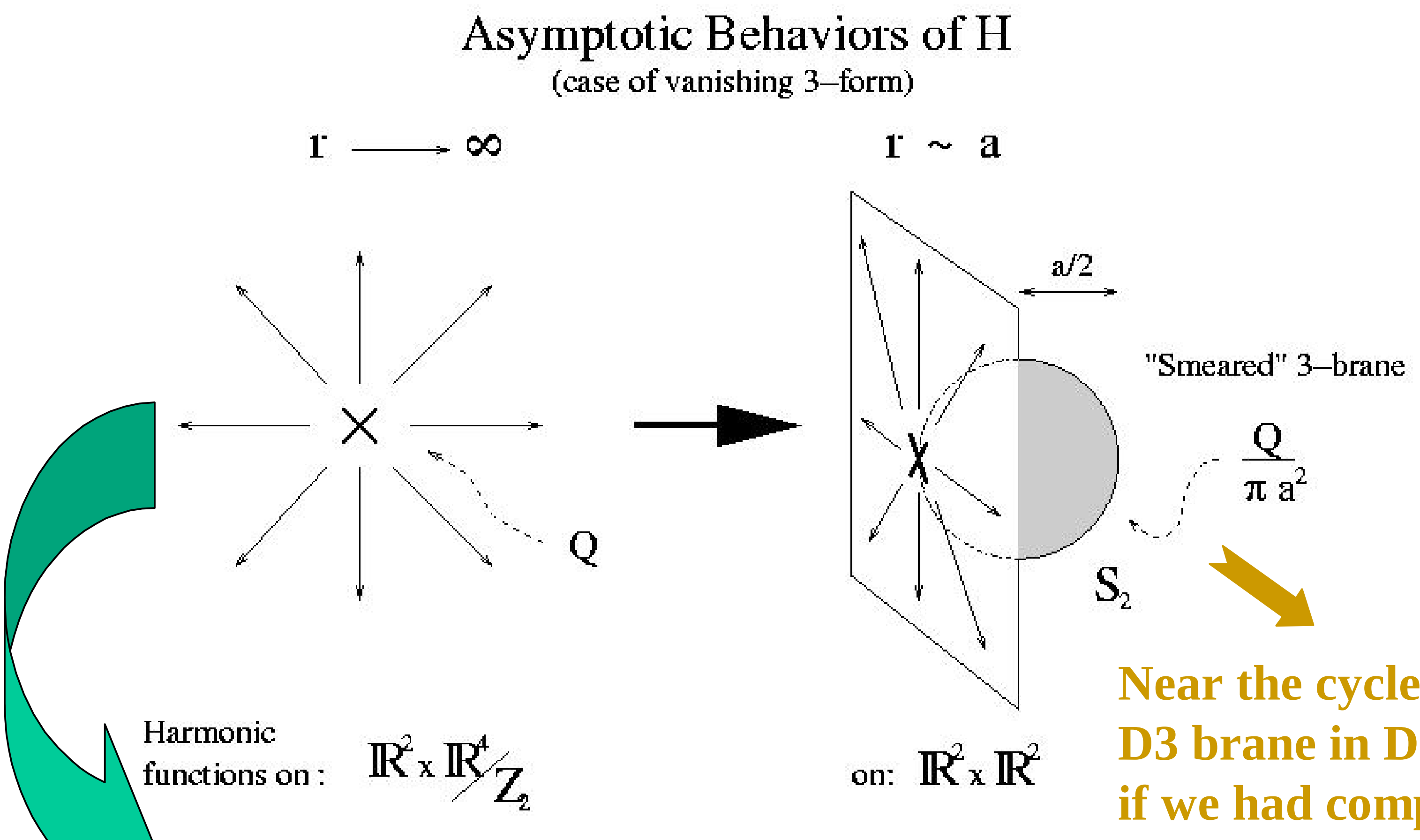
$$\begin{pmatrix} \rho < a \\ r \sim a \end{pmatrix} : \quad H(\rho, r) \sim \frac{Q}{2a^2}\frac{1}{\rho^2 + v^2} - \frac{1}{2\pi^2 a^4} K^2 \log\left(\frac{\rho}{\rho_0}\right)^2$$

See picture

P. Fré

SUPERGRAVITY AT 25
Stony Brook, 2001

# Interpolation and singularity




**Near the cycle we see a D3 brane in D=8. It is as if we had compactified sugra on an $S^2$**

**Far from the cycle we just see the charge of a normal D3 in D=10**

P. Fré

# The World Volume Action

$$
\begin{aligned}
\mathcal{A}_{D3-brane} = \int_{M_4} & \left[\Pi^{\underline{a}}_i V^{\underline{b}} \eta_{\underline{ab}} \eta^{i\ell_1} \wedge e^{\ell_2} \wedge e^{\ell_3} \wedge e^{\ell_4} \epsilon_{\ell_1\ell_2\ell_3\ell_4} \right. \\
& - \tfrac{1}{8} \left( \Pi^{\underline{a}}_i \Pi^{\underline{b}}_j \eta_{\underline{ab}} h^{ij} + 2 \left[ \det \left( h^{-1} + \mathcal{F} + \widetilde{\mathcal{F}} \right) \right]^{1/2} \right) e^{\ell_1} \wedge e^{\ell_2} \wedge e^{\ell_3} \wedge e^{\ell_4} \epsilon_{\ell_1\ell_2\ell_3\ell_4} \\
& + \mu\,(\phi, C_0)\, \mathcal{F}^{ij} \left( F^{[2]} + q_\Lambda A^\Lambda_{[2]} \right) \wedge e^\ell \wedge e^k \epsilon_{ij\ell k} \\
& \left. + \nu\,(\phi, C_0)\, \mathcal{F}^{ij} \left( F^{[2]} + q_\Lambda A^\Lambda_{[2]} \right) \wedge e_i \wedge e_j \right] \\
& + \int_{M_5} \left[ \mathcal{F}_5 + q_\Lambda H^\Lambda_{[3]} \wedge \epsilon_{\Gamma\Delta} q^\Gamma A^\Delta_{[2]} \right]
\end{aligned}
$$

Kappa supersymmetric: the fermions are hidden in the p—forms that are superforms

Red=Auxiliary fields

Brown=Sugra bckg fields

Blue=World volume fields

*The interactions of world—volume fields (fluctuations from vacuum values) feel all the bckg fields. Yet this action is source only for $C_4$. Where is the source of $A_{[2]}{}^{\Lambda}$ ?*

***This is the open question answering the which will shed light on the real nature of fractional branes***

# The End

Let us make an appointment for all supergravity people
to be back here for ***Supergravity @ 50***

***Such people are always running when someone watches..........***

P. Fré