Encounters with Supergravity

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Plan : Something new & Something Old

OLD
1) Intro : Why I failed to invent SUGRA

2) Killing Vectors & Killing Spinors

3) de Sitter, Anti de Sitter &
whatever happened to Twistor theory

NEW
4) Special Holonomy, Branes, Calibrations
& Cohomogeneity one metrics

$$\boxed{\text{INFINITESIMAL SUPERSYMMETRY}}$$

(Killing Spinor)

$$\vee$$

$$\boxed{\text{INFINITESIMAL ISOMETRY}}$$

(Killing Vector)

$$\delta_K g_{\mu\nu} = \mathcal{L}_K g_{\mu\nu} = 0 \iff \boxed{K_{\mu;\nu} + K_{\nu;\mu} = 0}$$

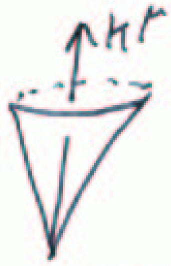
Killing's Eqns.

$$\delta_\epsilon \psi_\mu = 0 \iff \boxed{\begin{array}{l} \hat{\nabla}_\mu \epsilon = 0 \\ \hat{\nabla}_\mu \epsilon = \nabla_\mu \epsilon + E_\mu \epsilon \end{array}}$$

SUSY ALG $\Rightarrow$

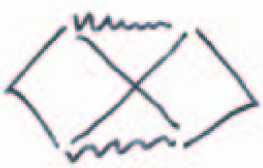
$$\boxed{K^\mu = \bar{\epsilon} \gamma^\mu \epsilon \quad \text{is Killing vector}}$$

Dirac's Lemma: K^μ is everywhere future directed timelike or null.

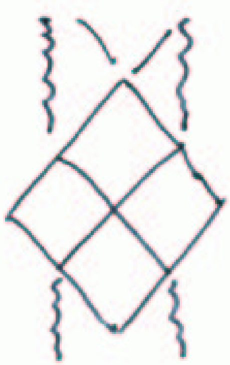


$$K^0 = \psi^\dagger \psi > 0 \quad \text{in all frames}$$

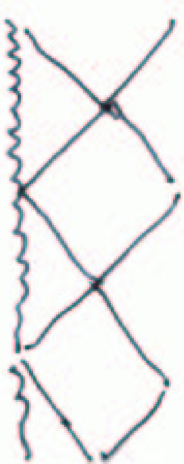
$\Rightarrow$ $\boxed{\text{Any BPS solution must be locally stationary \& may admit at most a } \underline{\text{degenerate}} \ \underline{\text{horizon}}}$



✗ $Q = 0$

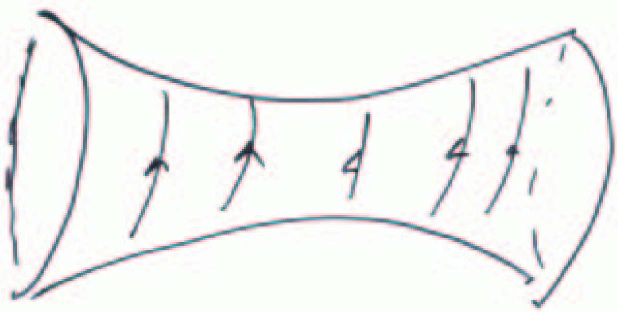


✗ $M > |Q|$



✓

De Sitter v. Anti-de Sitter



$\mathbb{E}^{4,1}$

$\mathbb{E}^{3,2}$

GLOBALLY STATIC

COSMOLOGICAL EVENT HORIZON

$$-(X^0)^2+(X^1)^2+(X^2)^2+(X^3)^2+(X^4)^2 = 1$$

✗

$$-(X^0)^2+(X^1)^2+(X^2)^2+(X^3)^2-(X^4)^2 = -1$$

✓

Wigner's Problem

$\forall\ H \in L(SO(4,1))\quad \exists\ g \in SO_0(4,1)$ s.t.

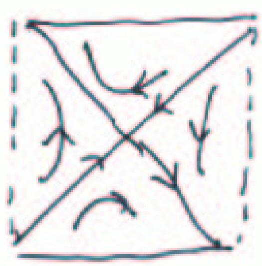
$$-H = g\, H\, g^{-1}$$

$\Rightarrow$ no positive energy $\Rightarrow$ no susy

$$X^0 = \sqrt{1-r^2}\,\sinh t \qquad X^i = r\,n^i\,,\quad n^i \in S^2$$
$$X^4 = \sqrt{1-r^2}\,\cosh t$$

$$ds^2 = -(1-r^2)\,dt^2 + \frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$L_{04} = \frac{\partial}{\partial t}$$



$$g = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{matrix} \in SO_0(4) \\ \in SO_0(4,1) \end{matrix}$$

g interchanges inside & outside of cosmological horizon.

Actually SUGRA $\overset{?}{\Rightarrow}$ STRONG ENERGY CONDITION

WHICH EXCLUDES ACCELERATION

Twistor Eqn

$$\boxed{(\nabla_\mu \gamma_\nu + \nabla_\nu \gamma_\mu)\,\epsilon = \frac{1}{n} g_{\mu\nu}\, \gamma^\lambda \nabla_\lambda \epsilon}$$

$$\nabla_\mu \epsilon = \frac{1}{2a} \gamma_\mu \epsilon$$

$$\boxed{\frac{1}{a^2} + \frac{\Lambda}{3} = 0}$$

$\Lambda < 0$; a real : $SO(3,2) \subset SO(4,2)$

$\Lambda > 0$; a pure imag : $SO(4,1) \subset SO(4,2)$

$\Lambda < 0 \longrightarrow \epsilon \gamma^\mu \epsilon$ Killing Vector

$\Lambda > 0$ $\quad \bar{\epsilon} \gamma \epsilon$ Conformal Killing Vector

$$\boxed{\text{Twistor Theory is contained within Conformal Supergravity.}}$$

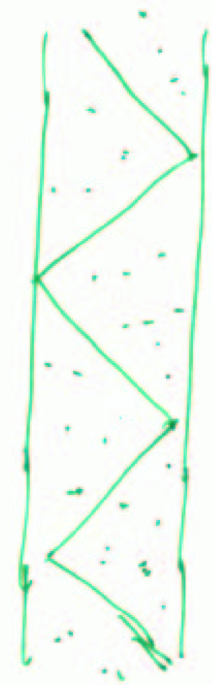
D3 - Branes or (M2 or M3)

$$ds^2 = -H^{-1/2}\left(-dt^2 + d\underline{x}_3^2\right) + H\, d\underline{y}_6^2$$

$$\nabla_y^2 H = 0$$

$$F_5 = d\left(\frac{dt \wedge dx_1 \wedge dx_2 \wedge dx_3}{H}\right) + *\left(d\left(\cdots \right)\right)$$

$$H = 1 + \frac{1}{|\underline{y}|^4}$$



No horizon geometry in $AdS_5 \times S^5$

Spatial interpolation between 2 maximal SUSY vacua

$\frac{1}{2}$ BPS.

M5 is like D3 non singular

M2 is like Reissner - Nordström : singular sources

Special Holonomy

Killing spinors of D3 are constructed from constant or parallel spinors of $\mathbb{E}^6$

($M2 \Rightarrow \mathbb{E}^8$, $M5 \rightarrow \mathbb{E}^5$ etc..)

Idea i) : replace $\mathbb{E}^n$ by curved mfd M^n admitting constant spinors

$$\nabla \epsilon = 0.$$

(ii) add additional NS & RR fluxes

$$d * F = i \, F_3 \wedge \bar{F}_3$$

$$\nabla^2 H = |F_3|^2$$

(Fractional branes)

need to find (M, g) & harmonic forms

Programme of wk. with Cvetic, Lu & Pope)

Special & Exceptional Holonomy

Berger 1955   M irreducible & not symmetric

$\mathrm{Hol}(M, g)$

$U(\frac{n}{2}) \subset SO(n)$: Kähler

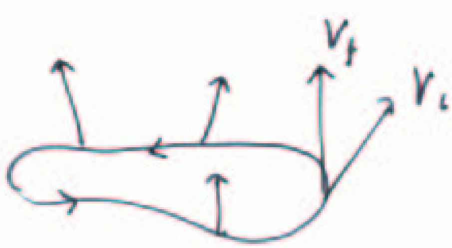
$SU(\frac{n}{2}) \subset SO(n)$: Calabi-Yau

$Sp(\frac{n}{4}) \subset SO(n)$: HyperKähler

$G_2 \subset SO(7)$: Exceptional

$Spin(7) \subset SO(8)$

$Sp(\frac{n}{4}) \cdot Sp(1) \subset SO(n)$: Quaternionic



$V_f = R V_i$

$R \subset \mathrm{Hol} \subseteq SO(n)$

Group generated by parallel transport.

$de + \theta \wedge e = 0$

$R = d\theta + \theta \wedge \theta$: Lie algebra valued 2-form values in $\mathrm{hol}(M, g)$

Holonomy gps which preserve one or more spinors

			SUSY's
$SU(\frac{n}{2})$	:	2 constant spinors	$N=2$
$Sp(\frac{n}{4})$	:	$n+1$ " "	$N=n+1$
G_2	:	$\underline{1}$ " "	$N=1$
$Spin(7)$	:	$\underline{1}$ " "	$N=1$

i) $\nabla_\mu \varepsilon = 0 \Rightarrow Ric = 0$.
[not true for Lorentz signature!]

ii) Hol is subgroup of $SO(n)$ leaving invariant a spinor

Eg. spinor rep of $SO(7)$: $\underline{8} = \underline{1} + \underline{7}$
$\uparrow$ preserved by $G_2 \subset SO(7)$

where in fact G_2 preserves the structure constants of the octonions.

Calibrating Forms

$$\bar{\epsilon}\,\gamma_{\mu_1\ldots\mu_p}\,\epsilon = \Phi_{\mu_1\ldots\mu_p} = \frac{1}{p!}\Phi_{\mu_1\ldots\mu_p}\,dx^{\mu_1}\wedge\ldots\wedge dx^{\mu_p}$$

$$\nabla\epsilon = 0 \quad\Rightarrow\quad \nabla\Phi = 0 \quad\Rightarrow\quad d\Phi = 0.$$

in fact $\qquad \Phi[\text{tangent plane}] \leq |\text{tangent plane}|$

now

$$\int_S \Phi = \int_{S'} \Phi$$

S is calibrated if $\quad \Phi|_S = \text{volume form}$

$$\Rightarrow \quad \int_S \Phi = \text{vol}(S) \leq \text{vol}(S')$$

S minimizes volume in its homology class.

SUSY cycles & Calibrating forms

BI action has K-symmetry

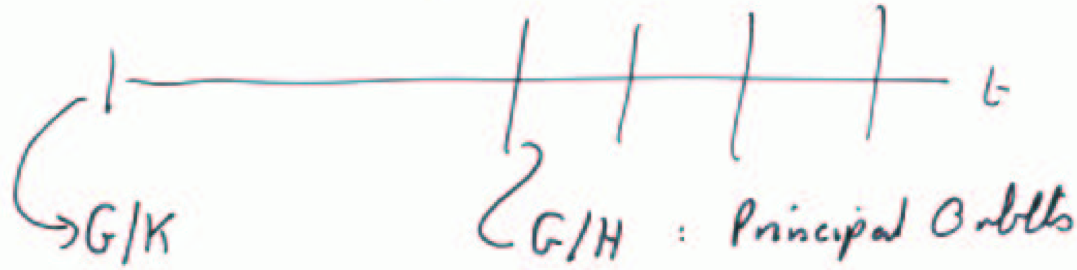
$\boxed{\text{K symmetry} \Rightarrow \text{SUSY} \Rightarrow \text{Calibrated}}$

$SU(\frac{n}{2})$	: ω (Kähler) Ω ← holomorphic 3-form ⇒ holomorphic cycles or SLAGS ; $\omega\|_S = 0$; $\mathrm{Re}\,\Omega\|_S = \mathrm{vol}$ SLAG's have dimension $\frac{n}{2}$
G_2 :	Φ ← associative 3-form — associative 3-fold $*\Phi$ — co-associative 4-fold
Spin(7)	: $\Phi = *\Phi$ self-dual 4-form Cayley ~~4-fold~~ 4-fold

Cohomogeneity one examples

G acts by isometries with co-dimension one principal orbits.

$$M = G/K \sqcup (\mathbb{R}_+ \times G/H)$$

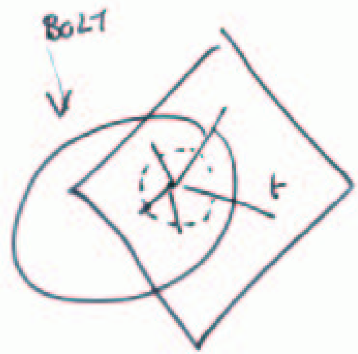


degenerate orbit
(BOLT)

$$K \supset H$$

$$K/H = S^d$$

Globally M^n is an $\mathbb{R}^{d+1}$ bundle over G/K
zero section



Example $M = T^* S^p$ $\quad G = SO(p+2)$

$K = SO(p+1)$

dim $M = 2p$ $\quad H = SO(p)$

$p = 3$ is "deformed conifold

Stenzel metrics

Einstein Eqns

$$ds^2 = dt^2 + g_{ab}(t)\, e^a(x) \otimes e^b(x)$$

$a = 1 \ldots \dim G/H$

e^a G covariant vielbein

Cartan-Maurer Forms $\quad \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$

$\leftarrow e^a$ are a basis for $\mathfrak{p}$

$g_{ab}(t)$ is a curve of G invariant metrics on G/H

configuration space "Superspace of Wheeler"

$$L = T - V \quad : \quad V$$

$\leftarrow$ Ricci scalar of G/H

$$G_{AB} \frac{dq^A}{dt} \frac{dq^B}{dt} - V(q^A)$$

DeWitt metric (Lorentzian signature!)

$$T = \mathrm{Tr}\, g^{-1}\dot{g}\, g^{-1}\dot{g} - \frac{1}{2}\left(\mathrm{Tr}\, g^{-1}\dot{g}\right)^2$$

2nd order ODES plus one constraint

$$\Rightarrow \quad \boxed{T + V = 0}$$

DIFFICULT TO SOLVE

Special Holonomy & Superpotentials

$$\boxed{\text{We find that imposing Special holonomy} \Rightarrow \text{1st order eqns which themselves imply the 2nd order eqns.}}$$

There are many ways of obtaining 1st order ode's but in all cases we have found a superpotential

$$V = -G^{AB}\frac{\partial W}{\partial q^A}\frac{\partial W}{\partial q^B}$$

$$\Rightarrow \quad \frac{dq^A}{dt} = G^{AB}\frac{\partial W}{\partial q^A}$$

In a number of cases these may be solved explicitly & completely (e.g. Spin(7) 3 function ansatz). In others they must be integrated numerically.

Behaviour nr. infinity

We typically encounter 2 types of asymptotics although other possibilities can certainly arise if we drop the cohomogeneity one assumption

(i) A C

Asymptotically conical $ds^2 = dt^2 + t^2 g^\infty(G/H)$

Ricci flat cone where $g^\infty(G/H)$ is Einstein metric

with Killing spinors

e.g.

Einstein Sasaki	5-dim	Calabi-Yau	6
Nearly Kähler	6-dim	G_2	7
Weak G_2	7-dim	Spin 7	8
Tri-Sasaki	– ~~8~~-dim	HK	8
Einstein Sasaki	– 7-dim	Calabi-Yau	8

In 8-dimensions we can have

Spin(7) $\cong$ Spin(7)	1	
SU(4) $\cong$ Spin(6)	2	
Sp(2) $\cong$ Spin(5)	3	
	SUSY's	

(14) Asymptotically Locally Conical ALC

This is more generic & favoured by the eqns.

$S^1 \times \dfrac{R \times (G/H \times SO(?))}{AC}$

a circle of constant length nr. ∞.

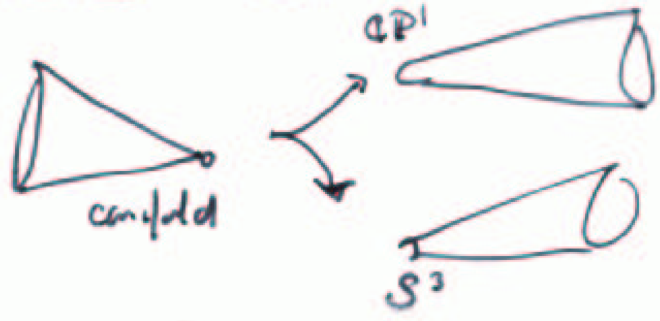
$\boxed{\text{These allow an M-Theory interpretation}}$

Typical moduli space

"Hausdorff - Gromov" limit in which a circle splits off as a product.

Calibrated BOLTS We typically find that the BOLT or degenerate orbit is a calibrated submfd.

Example 6-dim C.Y. cone over $T^{(1,1)} \cong S^1$ bdle over $S^2 \times S^2$



$O(-1)+O(-1)$ on CP^1 cplx line bundle.

stenzel: BOLT is SLAG.

Orientifold Planes & D6-Planes

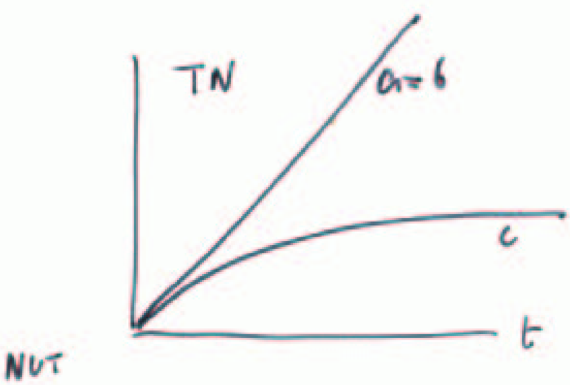
Taub-NUT v Atiyah-Hitchin

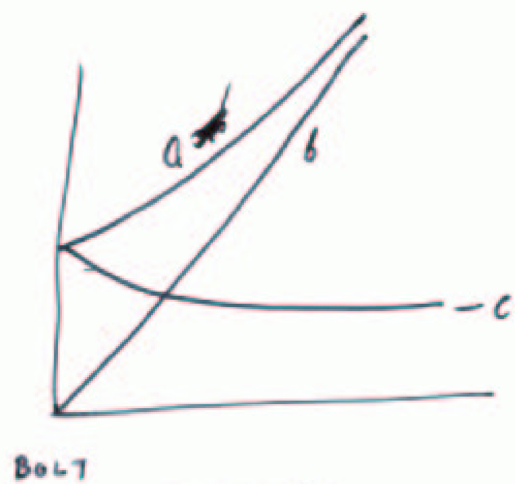
D6 O6

+ve mass -ve mass

$$ds^2 = -dt^2 + a^2\sigma_1^2 + b^2\sigma_2^2 + c^2\sigma_3^2$$

$d\sigma_1 = \sigma_2 \wedge \sigma_3$ etc.





+ve mass $\mathbb{R}^4$

$S(S^2)$

Asymptotically AH $\sim$ TN(-ve mass)/CP

$P : (r, \theta, \phi) \Rightarrow (r, \pi - \theta, \phi + \pi)$

$C : \psi \to -\psi$

$\leftarrow$ M-theory circle.

New Spin 7 examples

Old case

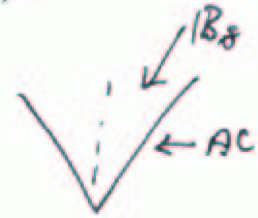
Bryant Salamon AC example on $S^+(S^4)$

$\uparrow$ chiral spinors

$G = SO(5)$ S^4 : Cayley 4-fold

New examples : i) ALC on $S^+(S^4)$

(explicit) (with parameters)



(ii) isolated ALC case on

$$\mathbb{R}^8 \Rightarrow \mathbb{A}_8$$

(iii) ALC example on ; $\mathbb{C}_8$

bundle over $\mathbb{CP}^2 \leftarrow S^2$

$$\downarrow_{S^4}$$

$$G/H \simeq SO(5)/SO(3) \cong S^7$$

wrapped orientifolds