The Construction of Supergravity Theories and their symmetries.

- Review the methods to construct a supergravity theory.

- show that D = 11 supergravity is a non-linear realization. hep-th/05270

- argue that D = 11, IIA and IIB supergravity theories have a large Kac-Moody symmetry (E_{11}). hep-th/0104081 0107181 0107209

Methods for Constructing Supergravity Theories

— Noether method using an action

— Noether method using the algebra

— On-shell superspace

— Dimensional reduction

— Gauging a space-time group
Chamseddine West.
MacDowell Mansouri

— Off-shell superspace Wess Zumino
Siegal.

The Noether method using an action

(...., Ferrara, Freedman, van Nieuwenhuizen)

Construction of off-shell supergravity.

Start from the linearized theory

$$h_{\mu\nu}\,,\ \Psi_{\mu\alpha}\,;\quad M,\ N,\ \underline{b_\mu}$$

The action

$$A^{(0)} = \int d^4x\,\{\,h\,\partial^2 h + \Psi\,\partial\,\Psi + M^2 + N^2 + \underline{b_\mu^2}\,\}$$

is invariant under

$$\delta h_{\mu\nu} = \varepsilon\,\gamma_{(\mu}\Psi_{\nu)}\,,\ \ldots\ldots$$

and

$$\delta\Psi_{\mu\alpha} = \partial_\mu\eta_\alpha\,.$$

Let $\varepsilon \longrightarrow \varepsilon(x^\mu)$ then

$$\delta A^{(} = \int d^4x\ \partial_\mu\varepsilon_\alpha\, j^{\mu\alpha}$$

$$j_{\mu\alpha} = (\partial h\Psi + \cdots)_{\mu\alpha}$$

An action invariant to order κ^0 is

$$A^{(1)} = A^{(0)} - \frac{\kappa}{2}\int d^4x\ \bar{\Psi}^{\mu\alpha} j_{\mu\alpha}\,.$$

provided $\eta_\alpha = \frac{2}{\kappa}\,\varepsilon(x)_\alpha$

Now add to action and transformations to gain invariance order by order in κ.

so

$$A = \int d^4x \left\{ \frac{e}{2\kappa^2} R - \frac{1}{2} \bar\psi_\mu R^\mu - \frac{e}{3} \left[M^2 + N^2 - b_\mu b^\mu \right] \right\}$$

invariant under

$$\delta \psi_\mu = \frac{2}{\kappa} D_\mu \varepsilon + i \gamma_5 \left(b_\mu - \frac{1}{3} \gamma_\mu \not{b}\right) \varepsilon$$
$$- \frac{1}{3} \gamma_\mu \left(M + i \gamma_5 N \right) \varepsilon$$

etc

Stelle West
Ferrara
van Nieuwenhuizen

Noether Method on the Algebra.

Construction of IIB supergravity

field content

$$e_\mu{}^a,\ A,\ A_{\mu\nu},\ A_{\mu_1\cdots\mu_4};\ \psi_{\mu\alpha},\ \lambda_\alpha$$ Nahm

The linearized equations of motion are invariant under the rigid supersymmetry transformation

$$\delta\lambda_\alpha = \gamma^\mu\partial_\mu A\,\varepsilon^* + \cdots$$

$$\delta A = \bar{\varepsilon}^*\lambda \qquad \text{etc.}$$

Let $\varepsilon \to \varepsilon(x)$ then

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\lambda_\alpha = \bar{\varepsilon}_1^*\gamma^\mu\varepsilon_2\partial_\mu\lambda_\alpha$$

$$+ \quad \gamma^\mu(\partial_\mu\bar{\varepsilon}_1^*\lambda)\varepsilon_2^* - (1\leftrightarrow 2)$$

provided $\not{\partial}\lambda = 0$

Closure to order κ^0 is given by

$$\delta\lambda_\alpha = \gamma^\mu(\partial_\mu A - \tfrac{\kappa}{2}\bar{\psi}_\mu\lambda)\varepsilon^*$$

provided $\delta\psi_{\mu\alpha} = \frac{2}{\kappa}\partial_\mu\varepsilon + \cdots$

Proceeding order by order in κ one finds the field transformations and equations of motion of IIB supergravity

Schwarz West

Schwarz

On-Shell Superspace.

The IIB supergravity content is

$$e_\mu{}^a \,,\ A \,,\ A_{\mu\nu} \,,\ A_{\mu_1 \dots \mu_4} \,,\ \psi_{\mu\,\alpha} \,,\ \lambda_\alpha$$

occurs as	$\partial\partial e$	∂A	$\partial_{[\mu_1} A_{\mu_2\mu_3]}$	$\partial_{[\mu_1} A_{\mu_2\dots\mu_5]}$	$\partial_{[\mu}\psi_{\nu]\alpha}$	λ_α
dimension	2	1	1	1	$3/2$	$\frac{1}{2}$
$U(1)$ charge	0	4	2	0	1	3

Superspace $x^\mu, \theta^\alpha, \theta^{\bar\alpha}$ contains the supervielbein $E_M{}^A$ and super connection $\Omega_M{}^{ab}$. The invariant objects are

$$T_{AB}{}^C = E_A{}^M \partial_M E_{B\}}{}^N E_N{}^C - \Omega_{\{AB\}}{}^C$$

and $R_{AB}{}^{cd}, \dots$

On grounds of $U(1)$ charge and dimension

$$T_{\alpha\beta}{}^c = 0 \,,\quad T_{\alpha\beta}{}^\gamma = 0 = T_{\alpha\bar\beta}{}^{\bar\gamma}, \dots$$

The Bianchi identities then solve for the torsions and curvatures in terms of a few superfields and give the equations of motion.

i.e $\quad T_{\alpha\beta}{}^{\gamma} \sim \delta_{(\alpha}{}^{\gamma} \Lambda_{\beta)}$

The Bianchi

$$D_a T_{\beta\gamma}{}^{\delta} + \quad \cdots \qquad = 0$$

$$\rightsquigarrow \quad \sigma^a D_a \Lambda + \cdots \qquad = 0$$

We identify

$$\Lambda_\alpha \big|_{\theta=0} = \lambda_\alpha(x^\mu) \,, \ldots$$

$$\rightsquigarrow \quad \sigma^a D_a \bar{\Lambda} = -\frac{i}{16.2} \sigma^{abc} \Lambda (\bar{\Lambda} \sigma_{abc} \Lambda) + \frac{i}{5.192} \sigma^{abcde} \bar{\Lambda} G_{abcde}$$

$$G_{abcde} = {}^{*}G_{abcde} - \bar{\Lambda} \sigma_{abcde} \Lambda$$

etc.

where

$$G_{\mu_1 \cdots \mu_5} = 5\, \partial_{[\mu_1} A_{\mu_2 \cdots \mu_5]} + 60 i \left(A_{[\mu_1 \mu_2} \partial_{\mu_3} A^{*}_{\mu_4 \mu_5} - c.c \right)$$
$$- 20\, \bar{\Psi}_{[a} \sigma_{bcd} \Psi_{e]}$$

Howe West.

Scalars in Supergravity Theories

The two scalars of $N=4$, $D=4$ supergravity belong to the coset $\dfrac{SU(1,1)}{U(1)}$ — Cremmer, Scherk, Ferrara.

The non-linear realisation of G with subgroup H; given $g(x^\mu) \in G$ the theory is invariant under (Callan, Coleman, Wess, Zumino)

$$g(x^\mu) \rightarrow \underset{\substack{\in G \\ \text{rigid}}}{g_0}\, g(x^\mu)\, \underset{\substack{\in H \\ \text{local.}}}{h(x^\mu)}$$

The Cartan forms transform as

$$\mathcal{V} \equiv g^{-1} dg \rightarrow h^{-1}\, \mathcal{V}\, h$$

Let the generators of G be T^a and $h_I \in H$ then we can use the local H invariance to choose

$$g(x^\mu) = \exp \phi^a(x^\mu) T_a$$

If G involves P^μ then

$$g = e^{x^\mu P_\mu} e^{\phi^a(x^\mu) T_a}.$$

Previous Results on Supergravity Symmetries

— The scalars in supergravity theories belong to cosets

— IIB has 2 scalars in $\frac{SU(1,1)}{U(1)}$ (Schwarz West)

— D = 11 supergravity reduced on a torus

	G	H	
D = 10, IIA	$SO(1,1)/\mathbb{Z}_2$	I	Campbell West, Giani Pernici, Huq Namazie
D = 9	$GL(2)$	$SO(2)$	
⋮			
D = 6	$E_5 \cong SO(5,5)$	$USp(4) \times USp(4)$	
D = 5	E_6	$USp(8)$	
D = 4	E_7	$SU(8)$	
D = 3	E_8	$SO(16)$	

(Cremmer, Julia, Julia, de Wit Nicolai)

— The gauge fields can also be included in a non-linear realization

Cremmer, Julia, Lu

Kac-Moody Algebras

Given a generalized Cartan matrix A_{ab} such that

(i) $A_{aa} = 2$

(ii) A_{ab} for $a \neq b$ are negative integers or zero

(iii) $A_{ab} = 0 \Leftrightarrow A_{ba} = 0$

and a set of Chevalley generators E_a, F_a, H_a which obey the Serre relations

$$[H_a, H_b] = 0 \; ; \; [H_a, E_b] = A_{ab} E_b$$

$$[H_a, F_b] = - A_{ab} F_b, \qquad [E_a, F_b] = \delta_{ab} H_a$$

and

$$\underbrace{[E_a, [E_a, \ldots \; [E_a}_{1 - A_{ba} \text{ factors}}, E_b] \cdots] = 0$$

plus relation for F_a's.

Then the Kac-Moody algebra is given by the multiple commutators

$$[E_{a_1}, [E_{a_2} \ldots, [E_{a_{n-1}}, E_{a_n}] \cdots]$$

plus multiple commutator of F_a's.

It is uniquely determined by A_{ab}

If A_{ab} is a positive definite matrix we get a finite dimensional semi-simple Lie algebra

Example $\quad A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

The Chevalley generators are

$$E_1 , E_2 , H_1 , H_2 , F_1 , F_2$$

The Serre relations are

$$[H_1, E_2] = - E_2 \quad \text{etc}$$

and

$$[E_1,[E_1,E_2]] = 0 \quad \text{etc.}$$

The resulting algebra is

$$\left.\begin{array}{ccc} E_1 , & E_2 , & [E_1, E_2] \\ (\alpha_1) & (\alpha_2) & (\alpha_1+\alpha_2) \end{array}\right\} \text{positive roots}$$

$$H_1 , H_2 \qquad \text{Cartan subalgebra}$$

$$\left.\begin{array}{ccc} F_1 , & F_2 , & [F_1, F_2] \\ (-\alpha_1) & (-\alpha_2) & (-\alpha_1-\alpha_2) \end{array}\right\} \text{negative roots}$$

ie $SU(3)$.

$D=11$ Supergravity as a Non-linear Realization

hep-th/2000-147

field $\quad e_\mu{}^a, \quad A_{a_1a_2a_3}, \quad A_{a_1\ldots a_6}$

generator $\quad K^a{}_b, \quad R^{a_1a_2a_3}, \quad R^{a_1\ldots a_6}.$

These obey an algebra G_{11} which contains $GL(11)$ $(K^a{}_b)$, P^a and

$$[K^a{}_b, R^{c_1\cdot c_3}] = \delta_b^{c_1} R^{a c_2 c_3} + \ldots$$

$$[R^{c_1\cdot c_3}, R^{c_4\cdot c_6}] = 2 R^{c_1\ldots c_6}$$

We consider the non-linear realization of G_{11} with subgroup $SO(1,10)$ ie

$$g = e^{x^\mu P_\mu} e^{h_a{}^b(x) K^a{}_b} e^{\{\frac{1}{3!} A_{c_1\cdot c_3} R^{c_1-c_3} + \frac{1}{6!} A_{c_1\ldots c_6} R^{c_1-c_6}\}}$$

• carry out simultaneous non-linear realization with conformal group $SO(2,11)$.

⤳ bosonic equations of motion of $D=11$ supergravity i.e

$$F_{a_1\cdot a_3} = \frac{1}{7!} \epsilon_{a_1\ldots\ldots a_{11}} F^{a_4\cdots a_{11}}$$

The Borel Subalgebra of E_7.

hep-th/0105270

Restricting the indices to $i = 5, \ldots, 11$ in G_{11} gives the subalgebra

$$D\ ,\ K^i{}_j\ ,\ R^{i_1 i_2 i_3}\ ,\ S_j = \frac{1}{6!}\epsilon_{j i_1 \cdots i_6} R^{i_1 \cdots i_6}$$

This is the Borel subalgebra of E_7

The adjoint of E_7 breaks to $sl(7)$ as

$$133 = 48\,(K^i{}_j) + 1\,(D) \qquad GL(7)$$

$$+\ 35\,(R^{i_1 \cdot i_3}) + \bar{7}\,(S_j) \qquad +\text{ve roots}$$

$$+\ \overline{35}\,(R^{i_1 \cdots i_3}) + 7\,(S^j) \qquad -\text{ve roots}$$

$\tilde{M}_{D=11}$ supergravity is invariant under the Borel subgroup of E_7.
If we increased the local subgroup so as to include $SU(8)$ in the restriction we would have a full E_7 symmetry.

Is D=11 Supergravity invariant under a Kac-Moody Symmetry?

G_{11} is not a Kac-Moody algebra, but we can

(i) use an alternative description of D=11 supergravity

(ii) enlarge the local symmetry.
($g \to g \circ g h$ so does not affect field content).

such that the resulting closure with the conformal algebra is a Kac-Moody algebra

The local subalgebra should be the one invariant under the Cartan involution.

$$g = \prod_a e^{H_a \phi_a} \prod_{\alpha \text{ +ve root}} e^{\phi_\alpha E_\alpha}$$

Identification of Kac-Moody Algebra. $\mathcal{G}$.

We split the generators in G_{11} into

$$G_{11}^{+} = \{ K^{a}{}_{b} \quad a<b, \qquad R^{c_1 \ldots c_3}, R^{c_1 \ldots c_6} \}$$

$$G^{0}_{11} = \{ H_a \equiv K^{a}{}_{a} - K^{a+1}{}_{a+1}, a = 1, \ldots 10, \quad D = \sum_{a} K^{a}{}_{a} \}.$$

The remaining generators are $K^{a}{}_{b}$, $a > b$ and are in the local subgroup

We expect that $G_{11}^{+} \subset \mathcal{G}_{+}$

Now G_{11}^{+} is generated by

$$E_a = K^{a}{}_{a+1}, \quad a = 1, \ldots, 10 \text{ and } E_{11} = R^{91011}$$

and identify these as the simple positive roots of $\mathcal{G}$.

We expect that $G_{11}^{0} \subset H$. and so a rank 11 algebra.

After choice of an appropriate basis for H to we can determine $\mathcal{G}$ from

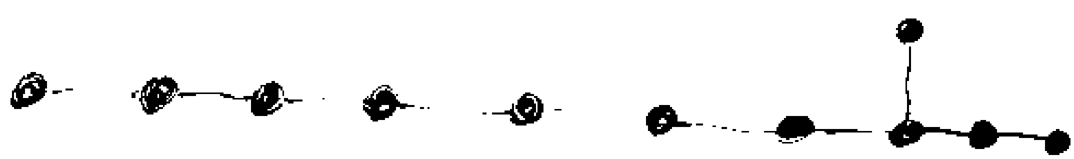
$$[H_a, E_b] = A_{ab} E_b.$$

Now E_a for $a = 1, \ldots, 10$ are the simple roots of A_{10} and E_a for $a = 5, \ldots, 11$ are the simple roots of E_7.

Given these embeddings the unique Kac-Moody algebra has a Cartan subalgebra

$$H_a \; , a = 1 \ldots, 10 \quad , H_{11} = K^9{}_9 + K^{10}{}_{10} + K^{11}{}_{11} - \frac{1}{3} D$$

and is E_{11}



G_{11} does not contain E_8. Under $SL(8)$ the adjoint of E_8 decomposes as

$$248 \to \underbrace{\underset{K^i{}_j,\ D}{63 + 1}}_{GL(8)} + \underbrace{\underset{R^{k_1 \cdot k_3}\quad \epsilon_{k_1 k_2 \cdots} R^{\cdots}\quad ?}{56 + \overline{28} + 8}}_{\text{positive roots}} + \underbrace{\overline{56} + 28 + \overline{8}}_{\text{negative roots}}$$

$i, j = 4, \ldots, 11$

where is the 8?

We can modify the G_{11} algebra by adding a new generator $R^{a_1 \cdots a_8, b}$

- gives Borel subalgebra of E_8

$$\underbrace{\begin{matrix} R^{c_1 \cdot c_3} & R^{c_1 \cdots c_6} \\ A_{c_1 \cdot c_3} & A_{c_1 \cdots c_6} \end{matrix}}_{\text{dual formulation of 3-form}} \quad ; \quad \underbrace{\begin{matrix} K^a{}_b & , \; R^{a_1 \cdots a_8, b} \\ h_a{}^b & h_{a_1 \cdots a_8, b} \end{matrix}}_{\text{dual formulation of gravity.}}$$

Conclusions

— We have argued that $D = 11$ supergravity is invariant under E_{11}

— A similar calculation implies that IIA and IIB are also E_{11} invariant.

($D = 11$ to IIA requires enlarged G_{11} algebra)

— The low energy effective action of the $D = 26$ closed bosonic string $(h_a{}^b, \phi, B_{a_1 a_2})$ has a corresponding Kac-Moody algebra of rank 27.

— requires a new formulation of gravity which should explain Geroch symmetries.

— Conjecture that M-theory is invariant under E_{11}