Jan. 12, 2009

Kontsevich, Wall-Crossing

Lattice Γ_u depending on parameter u
Local system.

$$\Omega_u : \Gamma_u \to \mathbb{Z}$$

jumps as we vary u

Appears in

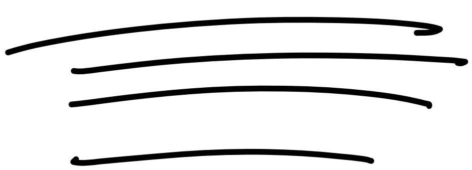
1. • Asymptotic geometry of SYZ collapsing CY
   KS 2003 Gross-Siebert
2. • Hyperkähler geometry of integrable systems.
3. • BPS counting in supersymmetric (2,2) $\hat{c} = 3$
4. • Stability in 3d CY categories and counting.

(1) X_ε $\quad \varepsilon \to 0$ approach cusp in moduli space

diameter $= 1$.

metrically $\qquad \dim_{\mathbb{C}} X_\varepsilon = n$

(i) small



B = real mfld
dim $= n$.

On B — singular set of cod $= 2$

Have integer affine structure

get Tropical limit of g_{ij}

real

Tropical CY   Real, affine str.
outside of sing.

Volume is const.

metric $= \partial_i \partial_j H$

$\det g_{ij} = \text{const.}$

Real Monge Ampere.

$$\begin{array}{c} (\mathbb{C}^*)^n \\ \downarrow \varepsilon \log|z_i| \\ B \supset U \subset \mathbb{R}^n \end{array}$$

Pull back of H: potential for Kähler metric on C.Y.

Mirror symmetry $\mathbb{Z}$ affine structure

Dual lattice — another affine str.

How to reconstruct back the CY mfld.

Collapse $\longrightarrow$ integral structure.

With Soibelman — construct a family of cplx C.Y.

Still open how to construct metric.

Want to include singularity

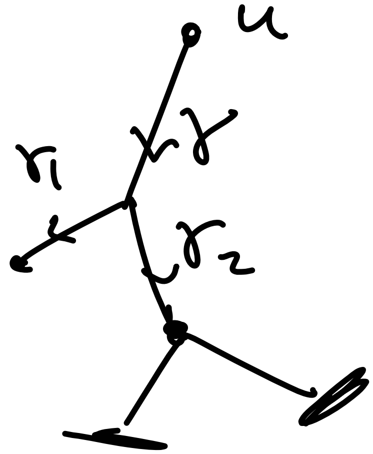
Naive family over B − sing.

Cut B by walls of real cod = 1 but ~~along~~ by walls make a change of coords.

Trees. Walls straight in affine.

$\bigcup$ walls = pts u in B s.t.

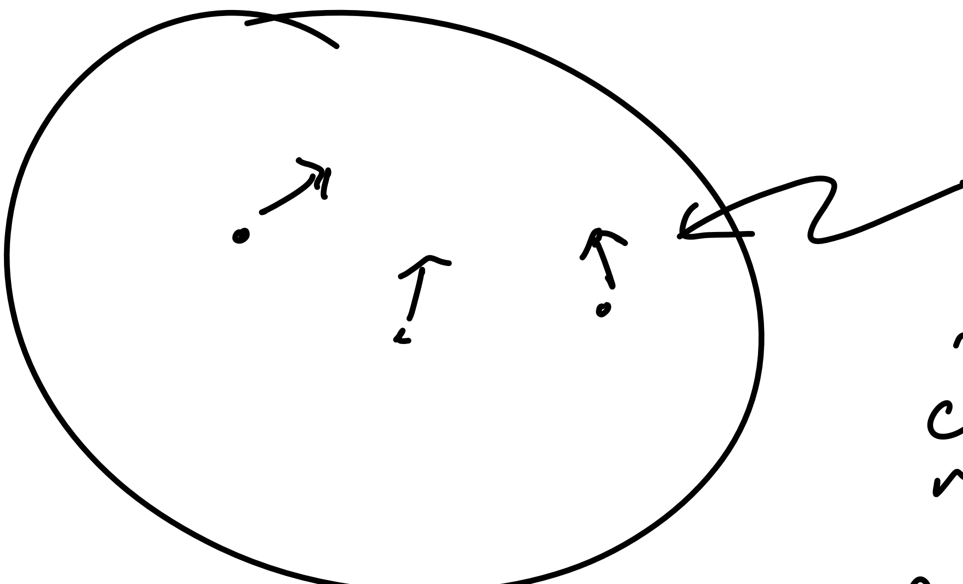
$\exists\ \gamma \in T_{\mathbb{Z},u}$ and a tree s.t. tree in $\mathcal{B}$ splits and ends



$\gamma = \gamma_1 + \gamma_2$

end @ singularities

K3 $\mathbb{P}^1$ w/ 24 pts.

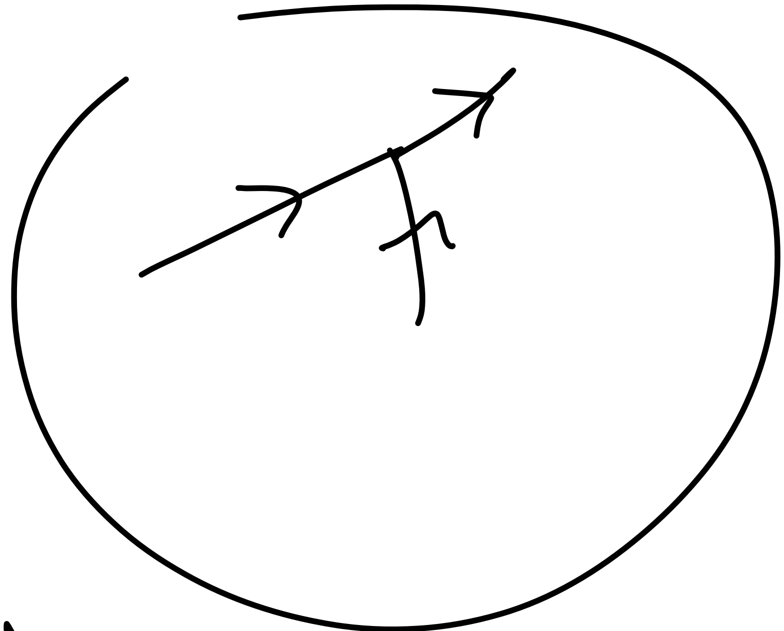


invt. direction under monodromy

choose multiplicity

Trees are densely filling up the $\mathbb{P}^1$!

Each tree has a multiplicity Ω



Change of coord's: All changes of coord's in two variables

$$x \longmapsto x \, F(x^p y^q, \varepsilon)^{q}$$

$$y \longmapsto y \, F(x^p y^q, \varepsilon)^{-q}$$

$p, q \sim$ direction of the tree

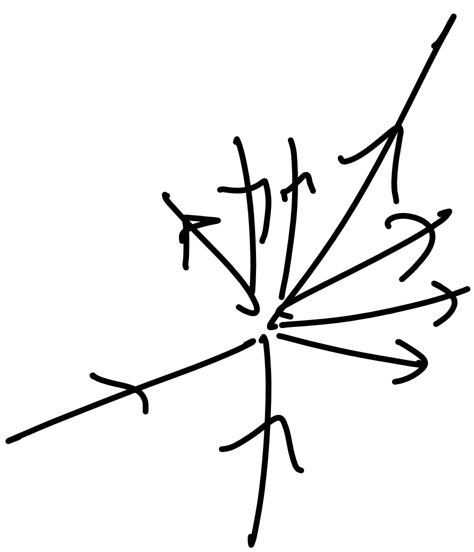
$$F(z) = 1 + \cdots$$

$\log|x|$, $\log|y|$ coord's on the base.

Construct change of coord's uniquely

$$x \to x$$
$$y \to y(1-x)$$

basic change of coord near singularity from Ooguri-Vafa metric.



how to construct the rest?

Holonomy of changes of coord's should be trivial.

$\Gamma_{p/q}$ $\quad p/q \in \mathbb{Q} \quad 0 \leq \quad < \infty$

$\gamma_0 \in \Gamma_0 \quad \gamma_\infty \in \Gamma_\infty$

in group of change of coords,
decompose

γ_∞ ---- $\gamma_{p/q}$ --- γ_0 $\quad$ $p/q \to$.

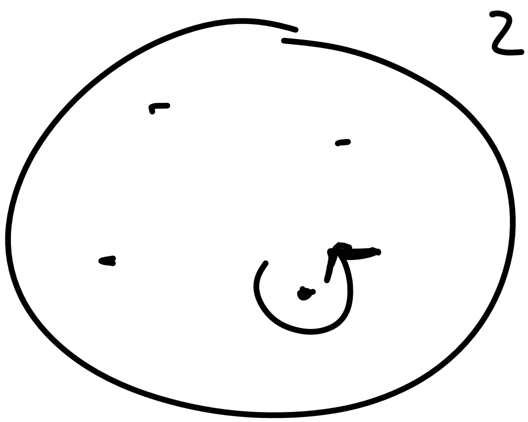
$\exists !$

ε – Shift ?

Canonical procedure ------

$$GL(n, \mathbb{Z}) \ltimes \mathbb{Z}^n$$

Get formal scheme over $\mathbb{Z}[[q]]$
Integral models of Calabi–Yau
like Tate Curve.



24 pts

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Bad Lagrangian. mfld on which
holo disks bound
projection of disks = trees.

Dense set of trees a problem
15 pages to modify so that near
a singular point get a disk.

This gives a <u>nonsingular</u>
complex manifold.

Can't really solve Monge Ampere
except for HK case.

<u>Integrable systems</u>

$$\begin{array}{c}\mathcal{Y}\\ \downarrow\\ \mathcal{B}\end{array}$$

holomorphic fibration
$\mathcal{Y}$ = holomorphic symplectic

fibers are Lagrangian principally polarized abelian varieties.

Write affine structure. Real cod 2 sing's.

$$\mathrm{Re}\left(\int_\gamma \omega^{2,0}\right) \quad \gamma \in H_1(\text{fiber})$$

= Closed real 1-forms

define affine str. on base

$\gamma_1, \ldots, \gamma_{2g}$ basis

Metric $\displaystyle \int_{\gamma_i} \omega^{2,0} \wedge \overline{\int_{\gamma_{g+i}} \omega^{2,0}}$ = Kähler form

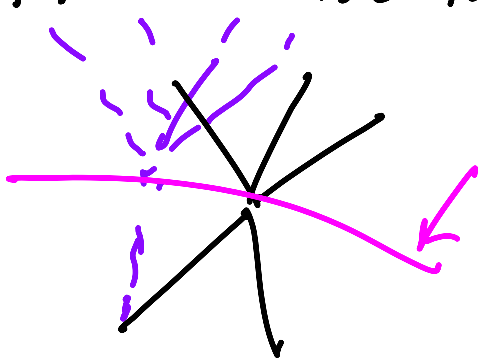
$\Rightarrow$ Riemannian metric on B.

1-parameter family λ

$$\mathrm{Re}\left(\lambda \int_\gamma \omega^{2,0}\right)$$

$Y_\lambda \qquad \lambda \in \mathbb{C}^* \qquad Y_0 = Y$

affine structure changes



So this is what occurs in Seiberg Witten theory

Over $\mathbb{CP}^1$ get twistor space

$\Longleftrightarrow$ HK metric.

So get 1-parameter family of HK metrics.

Formula for these metrics involves complicated decomposition of this group.

BPS Counting in Superconformal field theories $\hat{c} = 3$

Moduli of cplx str. on CY

$$\mathcal{M}_X \times \mathcal{M}_{X^\vee}$$

moduli of cplx str. on CY 3-fold

on dual 3-fold

Two triangulated categories A-branes
B-branes.

Has stable objects — "D-branes"

A, B lattices Γ^A, Γ^B

Even or Odd K-theory of C.Y.

$K^{top}_{even/odd}$

$\exists$ counting of D-branes.

Some integer #'s — depend on moduli space for another category

A-branes ~ special Lagrangian manifolds $\subset X$

moduli of cplx structures

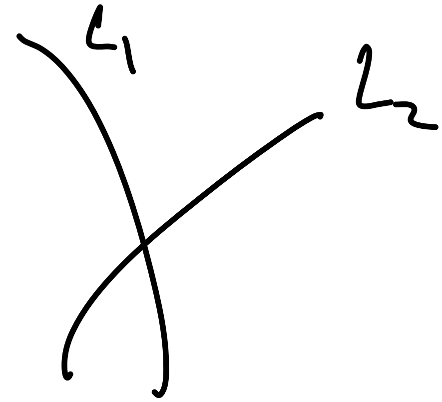
$[L] \in H_3(X, \mathbb{Z}) \longrightarrow$ 1 dim'l space $\int_L \Omega_X^{3,0} \in \mathbb{C}$

If two classes L_1, L_2 map to proportional - Physics - unstable

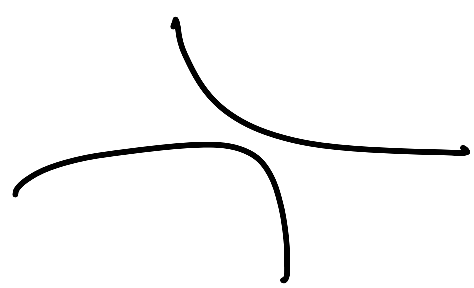
$\#$ special Lag. mflds not well-defined

$\int \Omega_X^{3,0}$ has const. phase.

$\gamma = \gamma_1 + \gamma_2$



Union is formally spec. Lag but singular.



exists on one side but not the other side.

Denef-Moore - Had formula in primitive vector case -

Soibelman + Kont. - general WCF.

Walls in moduli space $\mathcal{M}_X$

$$\begin{array}{l} \mathcal{L}_X \quad \text{choice of } \Omega^{3,0} \text{ form} \\ \big\downarrow \mathbb{C}^* \\ \mathcal{M}_X \end{array}$$

$$\mathcal{L}_X \longrightarrow \mathrm{Hom}\big(\underset{\underset{\displaystyle H^3(X,\mathbb{Z})}{\|}}{H_3(X,\mathbb{Z})}, \mathbb{C}\big)$$

Image is holomorphic Lagrangian cone

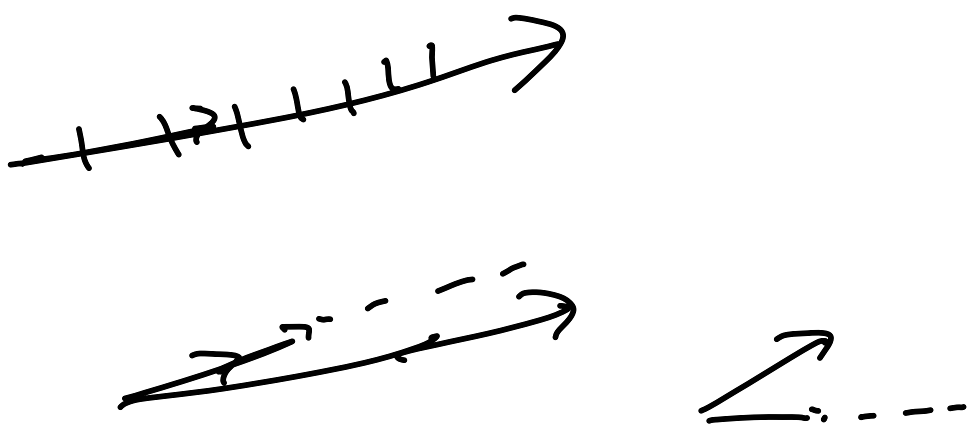
Walls: Ω s.t. $\exists\ \gamma_1, \gamma_2$ not proportional in $H_3 = \Gamma$ s.t.

$$\int_{\gamma_1} \Omega^{3,0} \in \mathbb{R} \int_{\gamma_2} \Omega^{3,0}$$

Want to write Ω outside walls

Generic point of the wall

$$\begin{array}{ccc} \mathbb{Z}^2 & \subset & H_3(\quad) \\ \downarrow \int \Omega^{3,0} & & \\ \mathbb{R}\cdot \text{const.} & \subset & \mathbb{C} \end{array}$$



Rough inequality: How to deduce from physics? Not any cohomology class can be represented

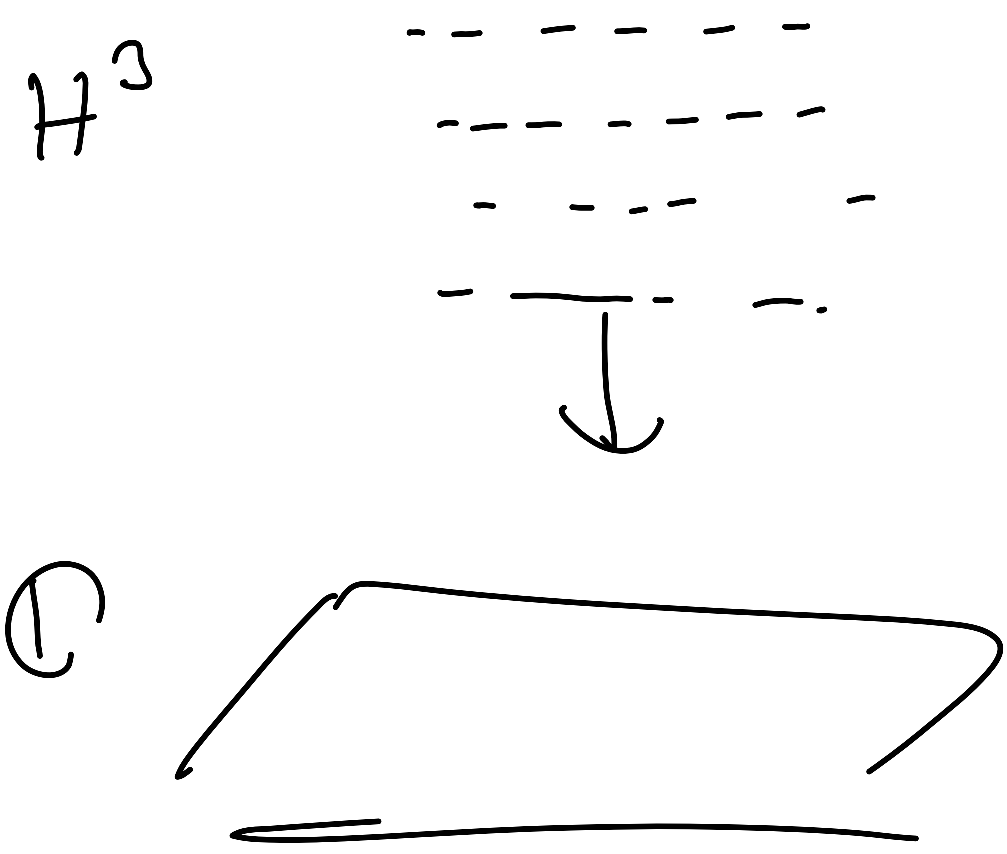
Choose any norm on $H_3(X, \mathbb{R})$

If a special Lag. mfld exists

$$\gamma = [L]$$

$$\|\gamma\| \leq C \cdot |Z(\gamma)|$$

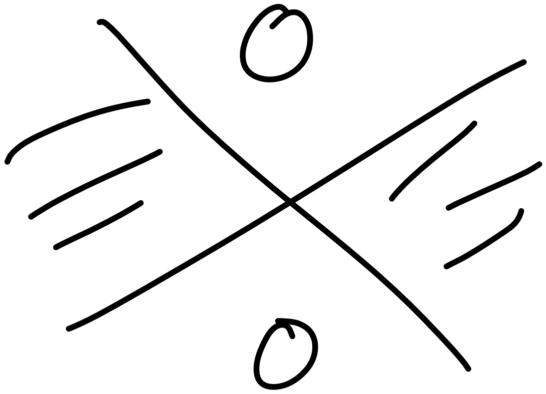
$$Z(\gamma) = \int_\gamma \Omega^{3,0}$$



No special Lag. mflds in big domain.

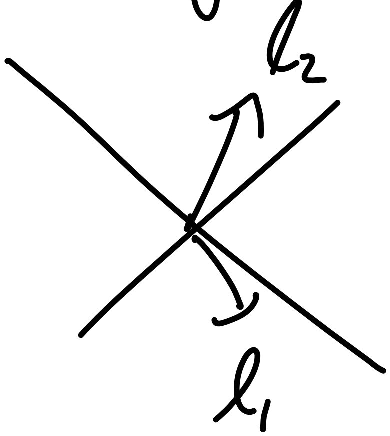
Special Lag. mflds are minimal volume.

True for general reasons.



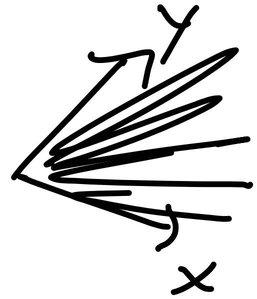
$Z = 0$ outside of two angle sectors

Choose generating vectors γ_1 γ_2



will open in two ways — two orientations.

$\mathbb{Z}^2$



$k \in \mathbb{Z}$

$k = \langle l_1, l_2 \rangle$

$a, b, \geq 0$

$$T_{a,b} \in \mathrm{Aut}\ \mathbb{Q}[[x, y]]$$

$$x \longrightarrow x\left(1-(-1)^{abk}x^a y^b\right)^{bk}$$

$$y \longrightarrow y\left(1-(-1)^{abk}x^a y^b\right)^{-ak}$$

$$\overrightarrow{\prod} T_{ab}^{\Omega_-(al_1+bl_2)} = \overleftarrow{\prod} T_{ab}^{\Omega_+(al_1+bl_2)}$$

Uniquely defines one set from another

$\Omega_+ \rightsquigarrow \Omega_-$ if Ω_+ is integral then so is Ω_-

Very nontrivial

Sign $(-1)^{ab}$ <u>absolutely crucial</u>

<u>Mirror Dual question:</u>

Bounded derived category
$D^b(\mathrm{Coh}\, X^\vee)$

cplx str and $\Omega^{3,0}$ on X

$\longrightarrow$ stability condition.

$\Omega(\gamma) =$ Euler characteristic of moduli of stable objects with
$\gamma \in K^{ev}_{top}(X^\vee)$

cplxs of
Vect. bdles on CY

Have chern classes AND

Secondary invts — in intermediate Jacobian

$$\frac{H^3(X^\vee, \mathbb{C})}{\text{Hodge filtration}} \to H_D(X^\vee) \to H^{ev}(X^\vee, \mathbb{Z}) \to 0$$

$H^{2,1} \oplus H^{3,0}$

$$\begin{array}{c} K_0^{alg}(X^\vee) \\ \downarrow \\ H^3_{\mathcal{D}}(X^\vee) \end{array}$$

Deligne cohomology = Lattice × Torus

Total space is holomorphic symplectic integrable system.

disconnected.

Classes of all holo. bundles for all complex structures $\Longrightarrow$ lots of Lagrangian submflds

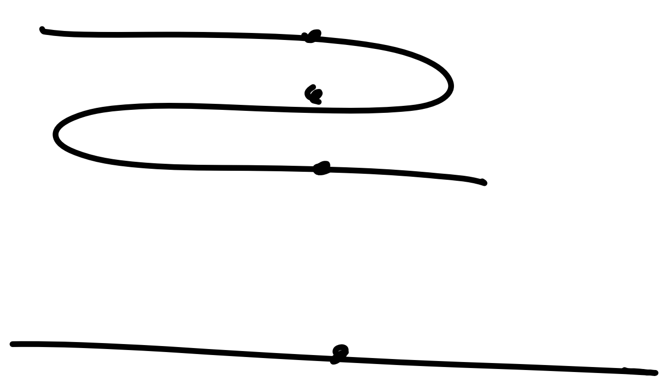
2 types: 1. projects to $\mathcal{L}_x$

$$\downarrow$$

$$\mathcal{M}_x$$

2. project to subvarieties.

Integer # associated to each component ...



fin. many points map to one

Holo CS action ~ superpotential.

Fix cplx str. on X

for each connected cpt of Lagrangian submf

$$\Omega(\gamma) = \sum_{\substack{\text{pts in} \\ J^3 \\ H^3_D \to \gamma}} \hat{\Omega}(pt)$$

$\exists$ other Lag. submflds that don't project.

Integer #'s ?

$\hat{\Omega}$: irred cpts of Lagrangian submflds of the int. Jacobian

3D CY Categories

Framework

$\mathcal{C}$ triangulated C.Y. 3d category
A_∞ category

$Ob\, \mathcal{C}$ $\forall\ \mathcal{E}, F$ $Hom^\bullet(\mathcal{E}, F) =$ fin. dual. $\mathbb{Z}$-graded space.

$$Hom(F, \mathcal{E}) \cong Hom(\mathcal{E}, F)^*[-3]$$

$$m_n : Hom(\mathcal{E}_0, \mathcal{E}_1) \otimes \cdots \otimes Hom(\mathcal{E}_{n-1}, \mathcal{E}_n) \longrightarrow Hom(\mathcal{E}_0, \mathcal{E}_n)[2-n]$$

$$(m_n(a_1, \ldots, a_n), a_{n+1}) \quad \mathbb{Z}/n+1$$

antisymmetric.

<u>Triang</u> $\mathcal{E} \rightsquigarrow \mathcal{E}[1]$

given a morphism can make a cone

$\Longrightarrow \exists$ short exact sequences

$$\mathcal{E} \to \,?\, \to \mathcal{F}$$

Triangulated is a property of A_∞.

$$\mathcal{C} \xrightarrow{\text{Yoneda}} \text{Functor}\,(\mathcal{C}^{op}, \text{complexes})$$

$H^0(\ldots)$ is a ~~t~~ triangulated category

D-branes do not form triang. cats
need to add by hand....

<u>Stability Theory</u>

$$K_0(\mathcal{C}) \to \mathbb{Z}^r \xrightarrow[\text{additive}]{Z} \mathbb{C}$$

$\uparrow$ huge

$\mathcal{C}^{ss}$ collection of semistable objects

$\forall \mathcal{E} \in \mathcal{C}^{ss} \quad Z(\mathcal{E}) \neq 0$

w/ chosen logarithm $\log Z(\mathcal{E})$

$$\log Z(\mathcal{E}[1]) = \log Z(\mathcal{E}) + i\pi$$

1.) $\operatorname{Im} \log Z(\mathcal{E}_1) > \operatorname{Im} \log Z(\mathcal{E}_2)$

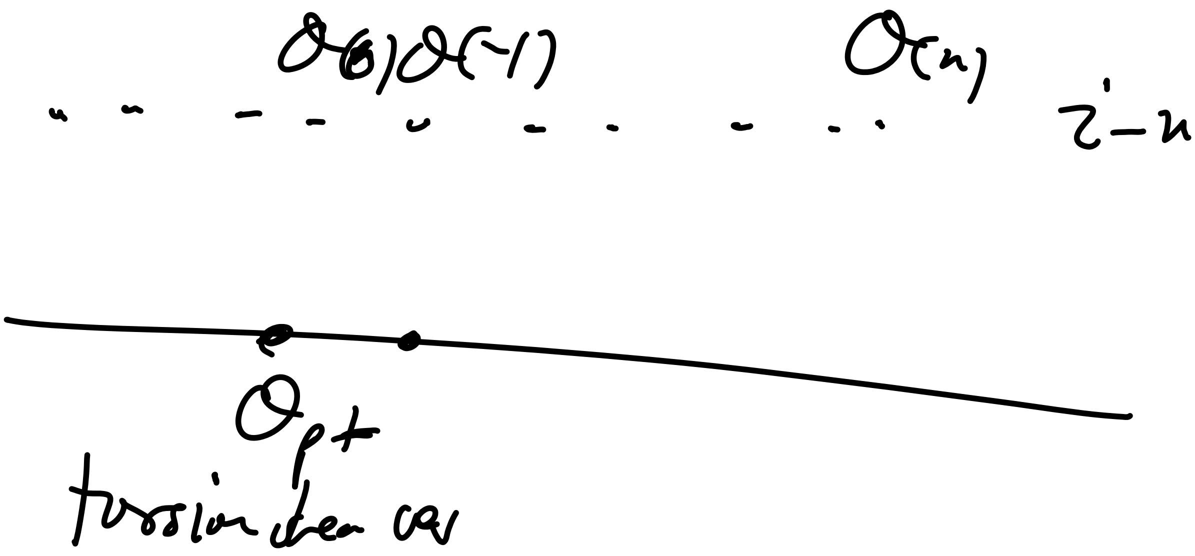
Then $\mathrm{Ext}^0(\mathcal{E}_1, \mathcal{E}_2) = 0$

and 2.) Any object $\mathcal{F}$ admits a canonical filtration.

$$0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \cdots \to \mathcal{F}_n = \mathcal{F}$$

Simple example:

$\mathcal{C} = D^b(\mathrm{Coh}\,\mathbb{P}^1)$



No map from torsion to nontorsion

$$C_{\geq 0} \supset C_{\geq 0}[-1]$$

Harder-Narasimhan stability

Put same inequality as for D brane

(Stronger than Bridgeland stab.)

Open nbd of space of Z's.

get some cplx mfld..

From physics get Lagrangian
Cone in this space of
Bridgeland conditions which have
arbitrary maps $Z: \mathbb{Z}^r \to \mathbb{C}$

$$\langle\ ,\ \rangle : \mathbb{Z}^r \otimes \mathbb{Z}^r \to \mathbb{Z}$$

$$\chi = \sum \mathrm{Ext}^i(\mathcal{E}, \mathcal{F})$$

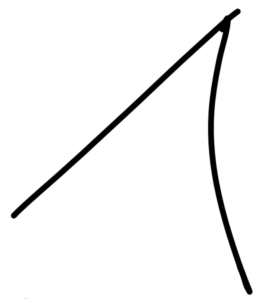
$$\mathcal{L} \subset \mathrm{Stab}$$

Lagrangian cone.
Can reconstruct hyperkähler mfld.
Lorentzian HK with
conformal homothety.

intermediate Jac $\longrightarrow \begin{pmatrix} \mathcal{L}_X \\ \downarrow \\ \mathcal{M}_X \end{pmatrix}$

Wall crossing limit is - - - -

Bundle with



germ @ ∞ of nonconical deformation of bundle of intermediate Jacobians - Close to Hypermultiplet moduli space.

Integrable system over $\mathcal{L}$.

Counting trees on moduli space $\mathcal{M}_X$

Attractor flows: $\gamma \in H_3(X, \mathbb{Z})$

cplx str. $\rightarrow \Omega^{3,0}$

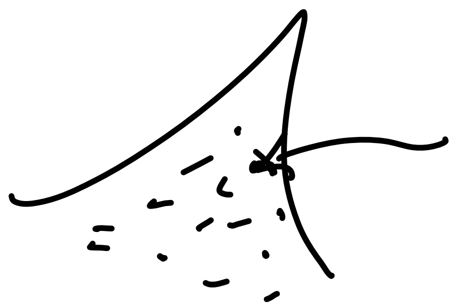
$$\frac{\left|\int_\gamma \Omega^{3,0}\right|^2}{\int |\Omega^{3,0}|^2}$$

gradient flows

end on attractor points

$$\mathcal{L} \xrightarrow[\mathrm{Re}]{} H^3(X, \mathbb{R})$$

integer points



Solution of wcf on whole space

arbitrary choice of Ω ~~at~~

at attractor points

$x \to (1-x)y$

~~Ω should vanish~~

Constraint