Topological defect theories &
framed bicategories of top. defects.

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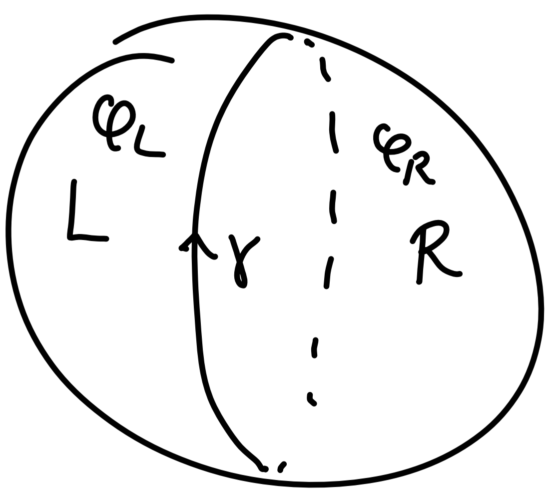
- Topological defects

first appearance — special class of defects in $N=2$ $SCFT_{2d}$



$g=0$

Lagrangian density

$$\mathcal{L} = \mathcal{L}_L + \mathcal{L}_R + \mathcal{L}_\gamma \left(\varphi_L|_\gamma, \varphi_R|_\gamma\right)$$

$$S = \int_R \mathcal{L}_R \, dv_R + \int_L \mathcal{L}_L \, dv_L + \int_\gamma ds_\gamma \mathcal{L}_\gamma$$

Flow to IR get CFT defects.

Topological defects – can move around.

So should be present in twisted

$\exists$ A, B – twisted topological defects

Topological.

Atiyah - Segal

Task generalize Atiyah–Segal formalism to "TFT_{2d} with defects"

$\boxed{\text{on R/S with no bdries}}$ ← only do this

on RS w bdries

A.S. Define 2D TFT | RS w/o bdries

as a strongly monoidal functor

preserves $\otimes$ up to isomorphism

Φ : Cob,

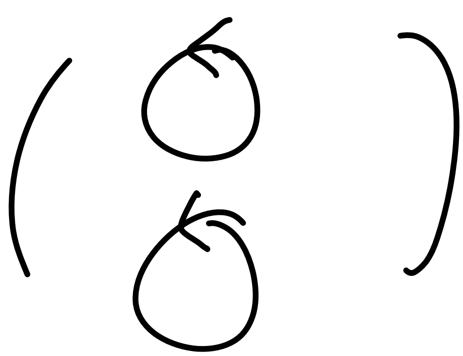
$$\Phi : (\text{Cob}, \quad) \longrightarrow \underbrace{(V, \otimes)}_{\text{symmetric monoidal category}}$$

V – valued

e.g. $V = \text{Vect}_k \quad k = \mathbb{C}$

Ob (Cob) = a orient. preserving diffeo class of finite disj. union of oriented $\bigcirc$

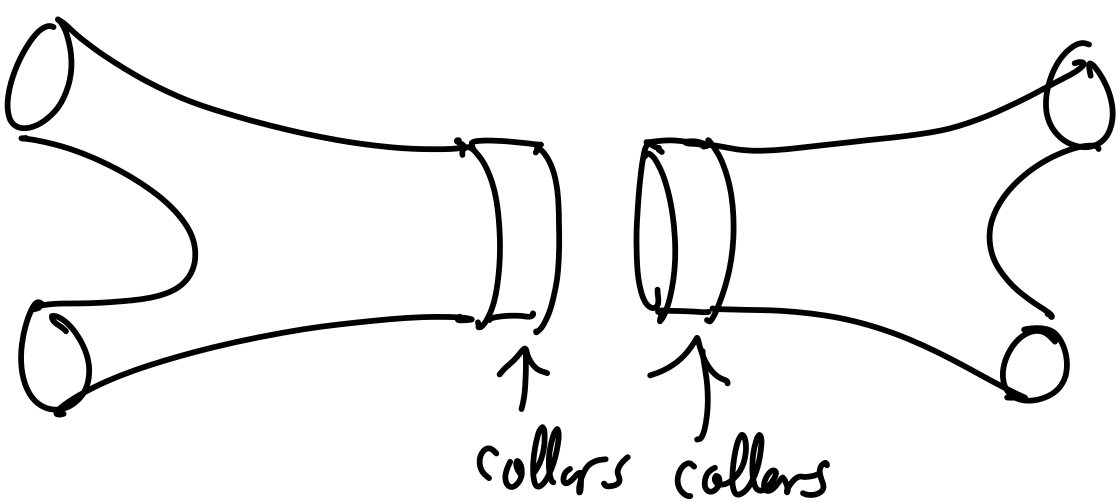
<u>oriented</u> everywhere.

e.g. 

Morphisms (Cob): 2D oriented cobordism $\Sigma : C_1 \to C_2$ is a 2d orientated cobordism $C_1 \to C_2$ (up to or. pres. diffeo.)

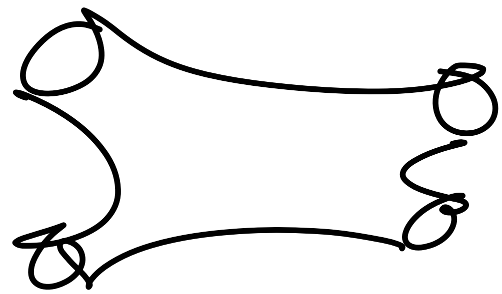
To do composition - associative only up to orient. pres. diffeos

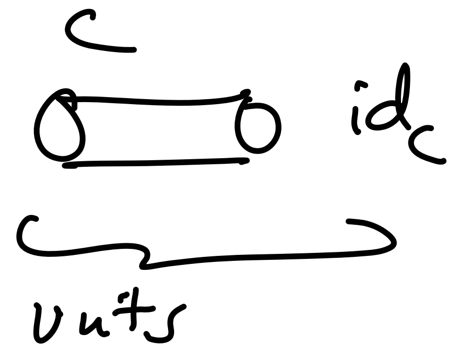
Composition: (sewing operation): (smooth-out) connected sum at the boundaries



because of callars it is not associative.

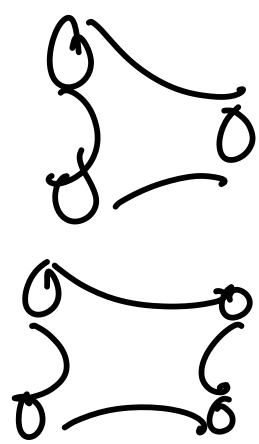
divide out by diffeos get associative composition





$(\mathrm{Cob}, \cdot) =$ Category

Monoidal structure = disjoint union



associative, symmetric so a strictly monoidal category

$(\mathrm{Cob}, \sqcup)$

Using pair of pants $\longrightarrow$ Data of commutative Frobenius alg.

How do we generalize this for defects?

Fusion of defects $\to$ decoration data

$\hookrightarrow$ from d.o.f. living on the defect. Call them levels. –

Fusion of defects: They need to carry an orientation.

So the data on the defect lives in a category.

<u>Defect cobordisms:</u>

(Decorated)

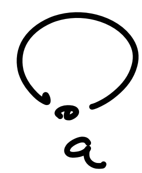
Decoration data is a "proto category" $\mathcal{E}$

A protocategory is the 1-step categorification of a unital magma ← set w/ operation semigroup

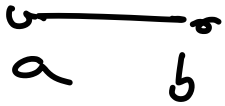
$\mathcal{E} : \left(Ob(\mathcal{E}), Mor(\mathcal{E}) \right) = \text{graph}$

$L, R \; Mor(\mathcal{E}) \longrightarrow Ob(\mathcal{E})$

$\forall \, a \in Ob(\mathcal{E}) \; \exists$ id_a



$Mor(\mathcal{E})(a,b) = Hom_{\mathcal{E}}(a,b)$



$Mor(\mathcal{E}) \times_{Ob(\mathcal{E})} Mor(\mathcal{E}) \longrightarrow Mor(\mathcal{E})$

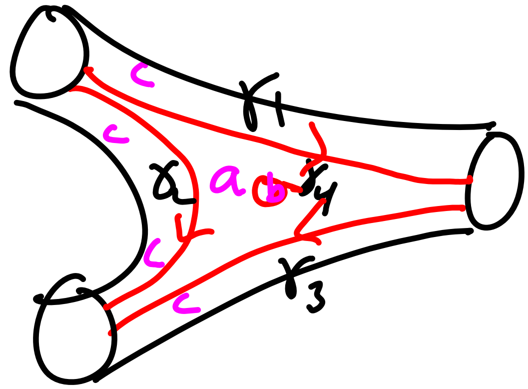
$a \xrightarrow{f} b \xrightarrow{g} c \implies a \xrightarrow{f} c$

Not necessarily associative category but with units. $id_a \circ f = f$

$g \circ id_a = g$

associativity is not required

e.g.

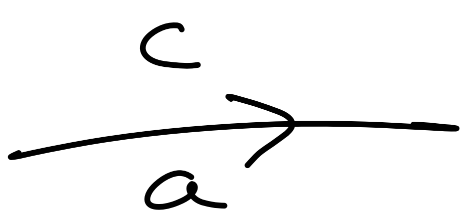


Objects ~ possible colors

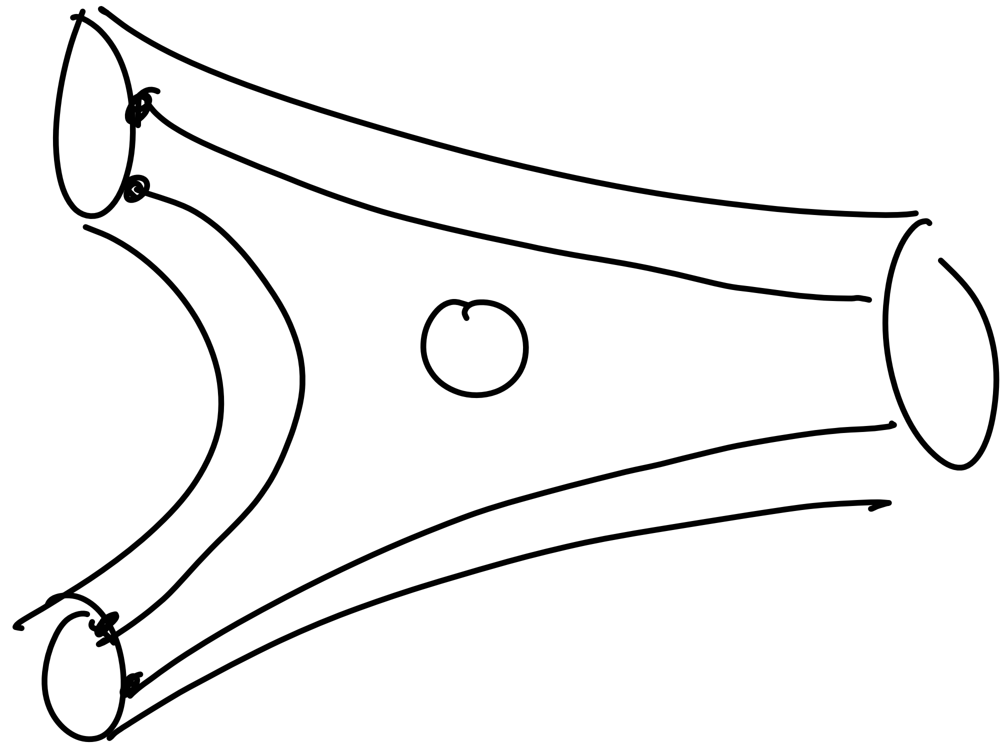
Cobordism — colors the components

$$\pi_0(\Sigma \setminus \Gamma) \longrightarrow Ob(\mathcal{E})$$

coloring map.



defect is decorated with a morphism $f \in Hom(c, a)$



Defect cobordism = t-uple

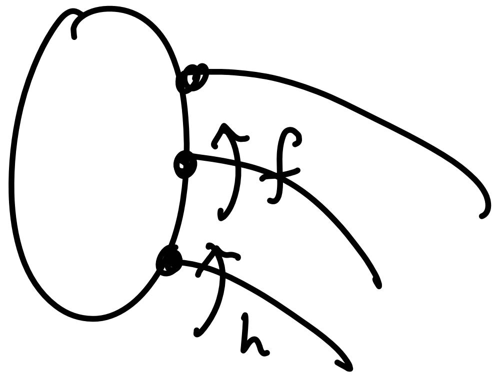
$$(\Sigma, \Gamma, \text{decoration})$$

$\Sigma \in \mathrm{Mor}(\mathrm{Cob})$

Γ = embedded oriented curve in Σ s.t. $\partial\Gamma \subset \partial\Sigma$

whose cpts $\gamma \in \pi_0(\Gamma)$ are called defect curves

Decorations (in Σ) as in the picture.



Object = disjoint union of circles with marked points with colors with morphisms.

Assume the theory has parity symmetry.

Assume $\exists$ an antiautomorphism

$f \quad \bar{\ } : \mathcal{E} \to \mathcal{E}$ involutive

$\bar{\ } : \mathrm{Mor}(\mathcal{E}) \to \mathrm{Mor}\,\mathcal{E}$

fixes nodes and reverses orientation of nodes.

$$\bar{\ } \circ \bar{\ } = id_{\mathcal{E}}$$

$$\overline{f \circ g} = \bar{g} \circ \bar{f} \quad \forall f, g \text{ composable arrows.}$$

Have to act nontrivial on the defect data.

Parity action on defect cobordism:

Invert orientation of Σ, of all T, act with $\overline{}$.

Composition $\mathrm{Cobd}_\mathcal{E}$: category of $\mathcal{E}$-def. cobordisms

$\mathrm{Mor}(\mathrm{Cobd}_\mathcal{E})$ as above

$\mathrm{Ob}(\mathrm{Cobd}_\mathcal{E})$ disj. unions of $\mathcal{E}$-def. circles

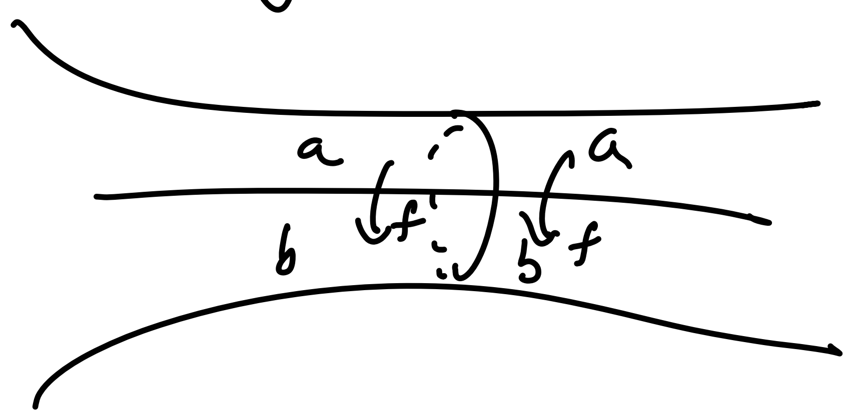
+ colors.

Up to orientation preserving diffeo.

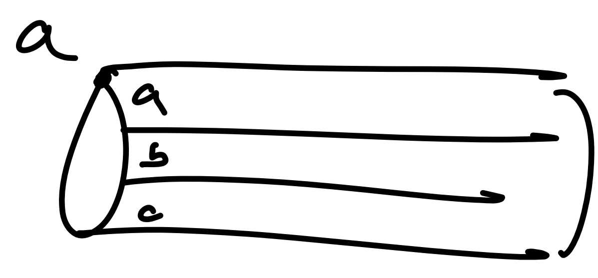
- smoothed connected sum

all above is up to orient. preserving diffeo and up to isotopies of f rel $\partial\Sigma$.

Sewing: Have to match everything



identity



Becomes a category —

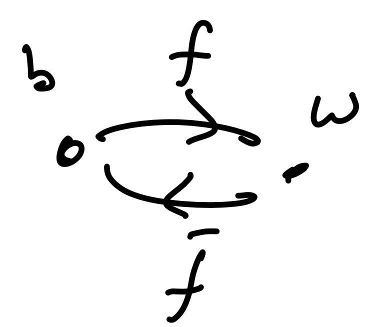
Strictly symmetric monoidal category

$(\mathrm{Cob}_{\mathcal{E}}, \sqcup)$

Def. An oriented 2D TFT w/ defects decorated in $\mathcal{E}$, and valued in $\mathcal{V}$ is a strong monoidal functor $\phi: (\mathrm{Cobd}_{\mathcal{E}}, \sqcup) \to \underbrace{(\mathcal{V}, \otimes)}$

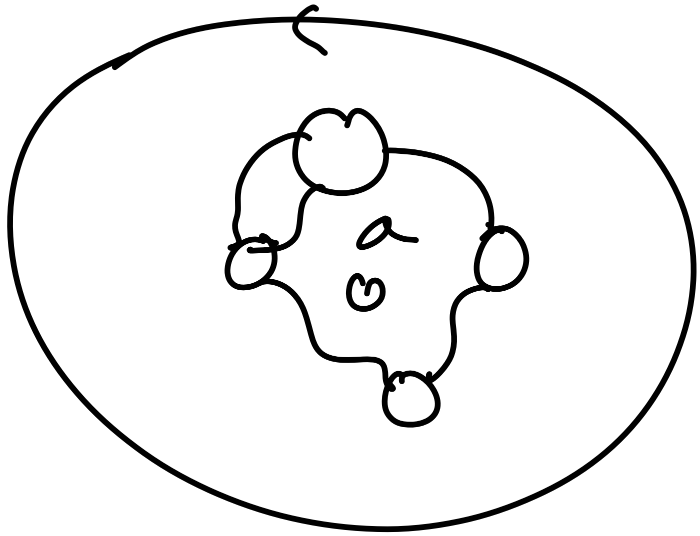
any symmetric monoidal category.

Examples give huge generalization of V. Jones planar algebra.



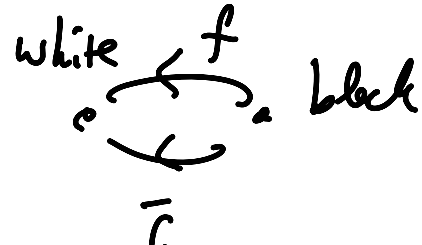
Restrict to cobordisms with single outgoing circle $\Longrightarrow$ operad.

$\mathrm{Cobd}_{\mathcal{E}}^{g=0,\ \pi_0(\partial \Sigma^{out})=0}$



Restriction to "multipants" is an enrichment of the little disks operad.

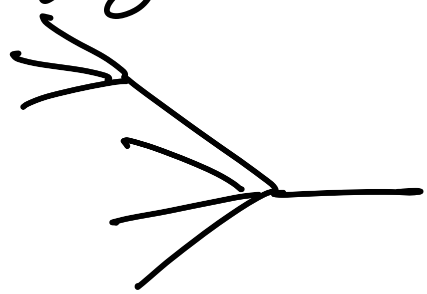
$\mathcal{E}$-decorated defect operad is an enrichment of the little disks operad

<u>Case</u> $\mathcal{E} =$ 

with $-$

gives planar operad of Jones.

Operad w/out symmetries is a multicategory



Can view ϕ as a multicategory

get a morphism

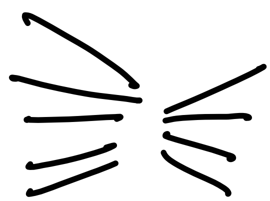
$$\bar{\phi} : \mathcal{P}_{\mathcal{E}} \longrightarrow \mathrm{Mult}(k)$$

Get an algebra over an operad w/ arbitrary # of colors and w/out symmetry. $\Longrightarrow$

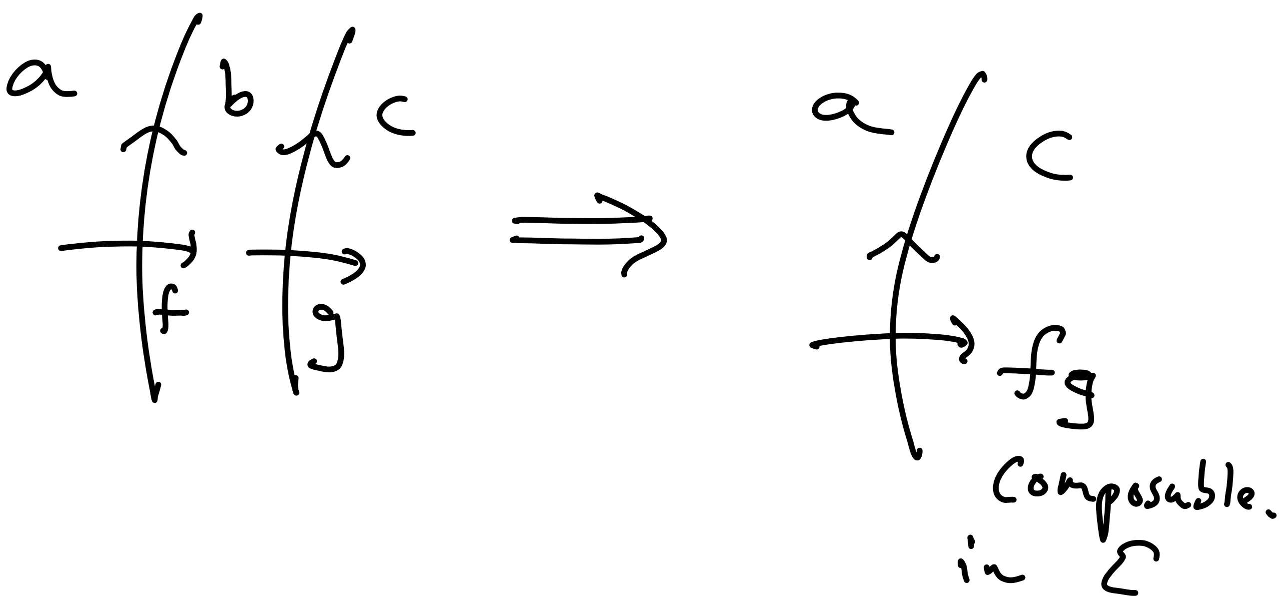
$\mathcal{E}$-decorated algebra.

Related to poly-bicategories

Theory ϕ has fusion of defects iff a certain associated poly-bicategory is <u>representable</u>.



poly-bicategory - - - - - -



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How to apply this.

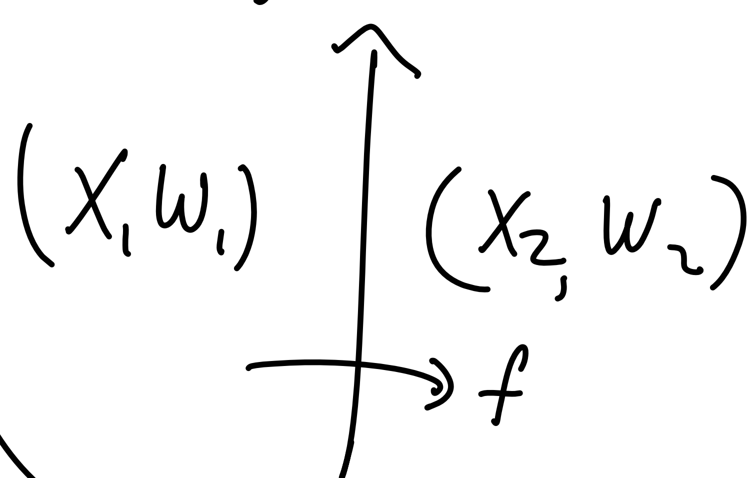
Main examples

<u>Ex1</u>: B-type topol. LG models with defects

Put together matrix factorizations

$f \in HMF\left((X_1, W_1), (X_2, W_2) \right)$

Homotopy Matrix factorization category



$$\underset{def}{=} HMF\left(X_1 \times X_2, -W_1 \otimes 1 + 1 \otimes W_2 \right)$$

Ex2: Caldararu + ...

Fourier-Mukai in CY case

B-twisted NLσM.