Nagao

arXiv 0809.2992 w/ H. Nakajima
2994

$A = (\overset{\text{quiver }(I,H)}{Q}, \overset{\text{potential}}{\omega})$: A quiver w/ a super-potential (3-dim CY category of reps)

+ stability parameter

$\longrightarrow$ moduli spaces of framed reps of A $\longrightarrow$ counting invariants

generalized DT theory for A.

- Wall-crossing phenomenon.

e.g. conifold

Q

$$\begin{tikzcd} 0 \arrow[r, bend left, "a_1"] \arrow[r, bend left=60, "a_2"'] & 1 \arrow[l, bend left, "b_1"'] \arrow[l, bend left=60, "b_2"] \end{tikzcd}$$

$$w = a_1 b_1 a_2 b_2 - a_1 b_2 a_2 b_1$$

$$Y = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \xrightarrow{\pi} \mathbb{P}^1$$

$$P = \mathcal{O}_Y \oplus \mathcal{L} \qquad \mathcal{L} = \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$$

A = Path algebra with relations

$\cong \mathrm{End}(P)$

$$\begin{array}{ccc} D^b(\mathrm{Coh}\, Y) & \xrightarrow[R\mathrm{Hom}(P,-)]{\sim} & D^b(\mathrm{mod}\, A) \\ \cup & & \cup \\ \mathrm{Coh}\, Y & & \mathrm{mod}\, A \end{array}$$

$\hookrightarrow$ abelian categories, do not coincide under this equivalence.

Original DT was on Coh Y

Today we are counting objects in mod A.

$i \in I$ = vertices

P_i = Projective A-module

$P_i = \{$ linear combination of paths starting from $i\}$ / relations

(A = path algebra)

$$A = \bigoplus_{i \in I} P_i$$

$V \in \mathrm{mod}_{\text{fin. dim}} A = \mathrm{mod}_f A$

Consider moduli space of pairs

$$\left(P_i \xrightarrow{s} V \right) \quad \text{for a fixed } i$$

motivation.

$$D^b \operatorname{Coh} Y \longrightarrow D^b \operatorname{mod} A$$

$$\mathcal{O}_Y \longleftrightarrow P_0$$

DT, PT $\rightarrow$ pairs $(\mathcal{O}_Y \to F)$ $\rightsquigarrow$
stab.

<u>Def</u> $\zeta \in \mathbb{R}^I$ stability parameters

$(P_i \xrightarrow{s} V)$ is ζ-(semi)stable

$\overset{\text{def}}{\Longleftrightarrow}$ (1) $\forall$ $0 \neq V' \subseteq V$ A submodule

$$\zeta \cdot \underline{\dim}\, V' \underset{(\leq)}{<} 0$$

An A-module can be decomposed

$$V = \bigoplus_{i \in I} V_i \qquad \underline{\dim}\, V = (\ldots \dim V_i \ldots)$$

(2.) $\forall$ $V' \subsetneq V$ A submodule

s.t.

$$\begin{tikzcd} V' \arrow[r, hook] & V \\ & P_i \arrow[ul, dashed, "\exists"] \arrow[u, "s"'] \end{tikzcd}$$

$$\zeta \cdot \underline{\dim}\, V' \underset{(\leq)}{<} \zeta \cdot \dim V$$

$$\Longrightarrow \quad \mathbb{V} \in \mathbb{Z}_{\geq 0}^{I}$$

$$\mathcal{M}^{\zeta}_{A,P_i}(\mathbb{V}) := \left\{ (P_i \xrightarrow{s} V) \;\middle|\; \begin{array}{l} V \in \mathrm{mod}_f A \\ \underline{\dim}\, V = \mathbb{V} \\ \zeta\text{-semistable} \end{array} \right\} \Big/ \sim$$

$\mathbb{C}$

Constructed GIT quotient so this is a well-defined scheme

$$Z^{\zeta}_{A,P_i}(\vec{q}) = \sum_{\mathbb{V}} \deg \left[\mathcal{M}^{\zeta}_{A,P_i}(\mathbb{V})\right]^{\mathrm{Vir}} q^{\mathbb{V}}$$

For generic ζ

Moduli spaces noncompact, so define using Behrend's function.

Most examples derived from brane tilings so $\deg = (-1)^{\cdots}$ Eulercharacter

e.g.

(1) $\zeta_{triv} = (\zeta_{triv}^{i}) \in \mathbb{R}^{I}$

$\zeta_{triv}^{i} > 0 \quad \forall i \in I$

$(P_i \xrightarrow{s} V)$ is ζ_{triv}-stable

$\Longleftrightarrow V = 0 \Rightarrow$ trivial generating function.

(2.) $\zeta_{cyclic} = (\zeta_{cyclic}^{i}) \quad \zeta_{cyclic}^{i} < 0$

$(P_i \xrightarrow{s} V)$ is stable iff

s is surjective.

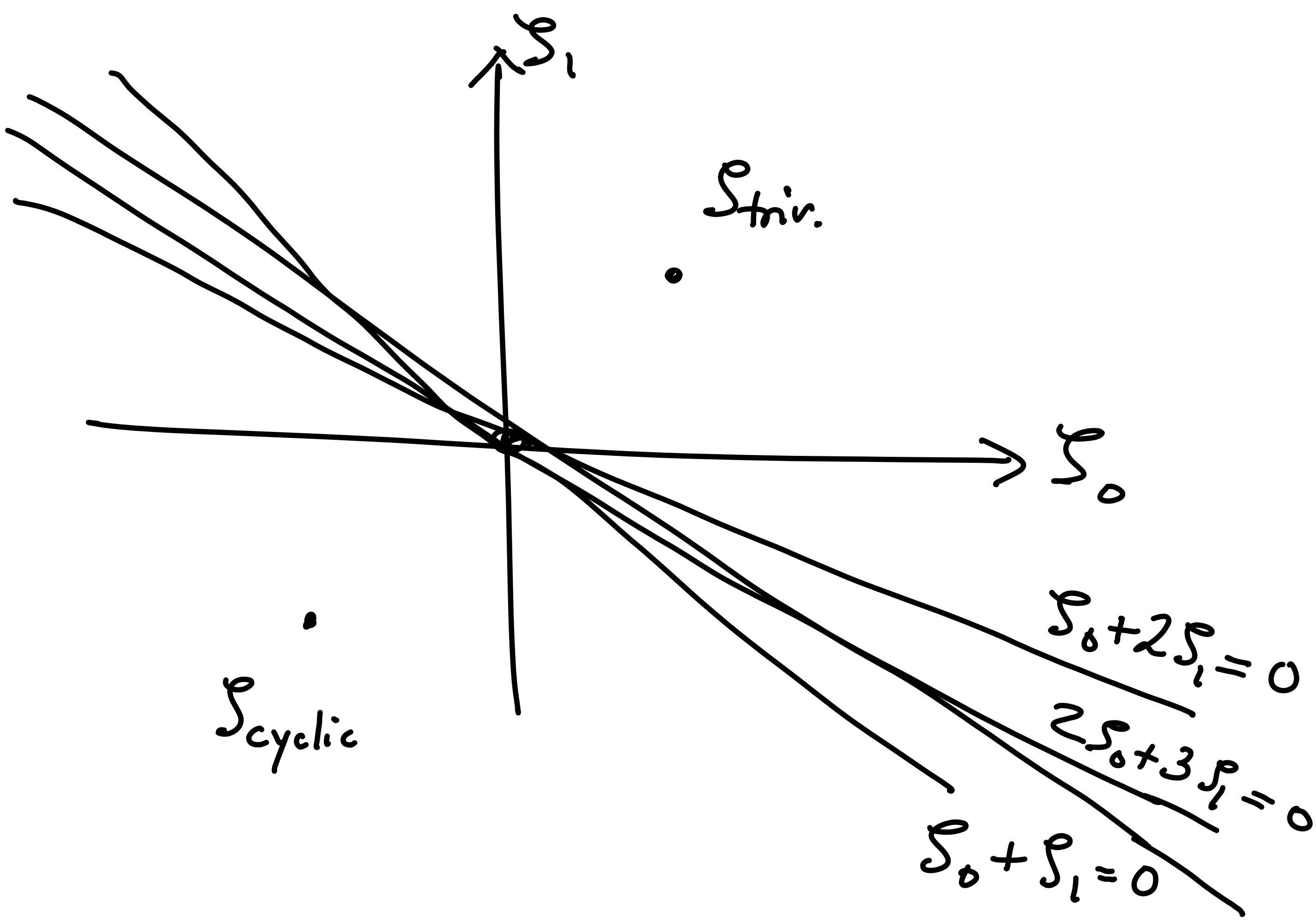
(1) $\forall V' \subseteq V \quad \zeta . \dim V' \leq 0.$

(2) ....

(2) $\Longrightarrow$ Szendrői's NCDT

(1) trivial gen. fn. $\Longrightarrow$ now study wall-crossing for all chambers.

e.g. (conifold) Parameter space is 2-dimensional.

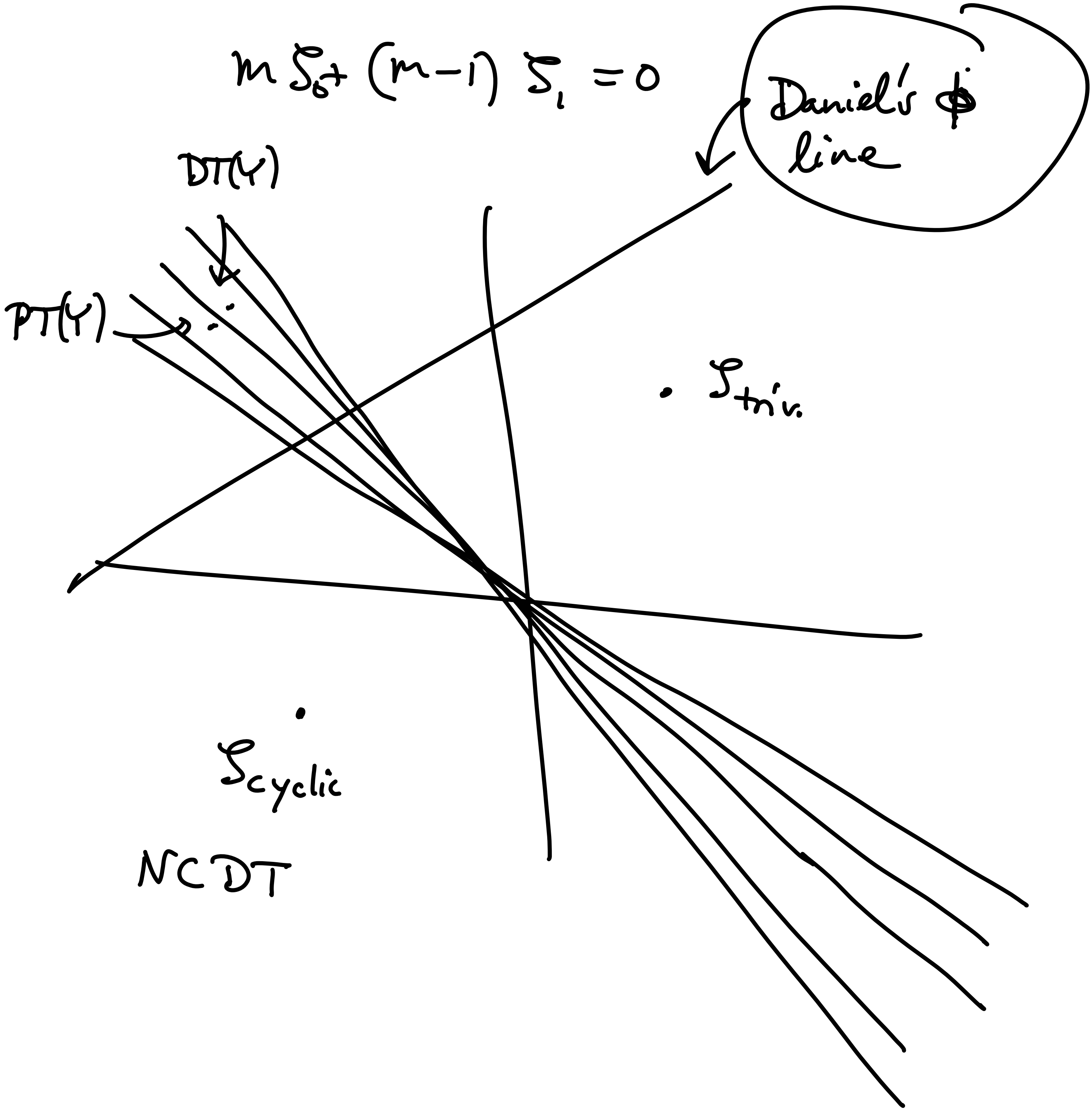


$$m\zeta_0 + (m+1)\zeta_1 = 0 \qquad m \in \mathbb{Z}$$

asymptotes to $\zeta_0 + \zeta_1 = 0$

Also have walls

$$m\zeta_0 + (m-1)\zeta_1 = 0$$



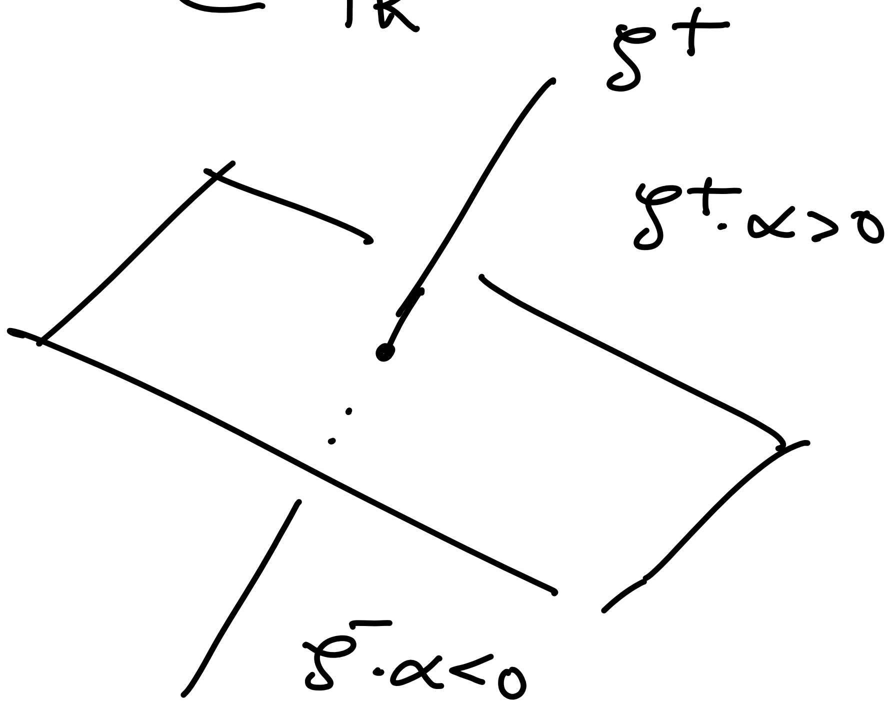
$$Z^{NCDT} = Z^{DT}(Y) \cdot Z^{PT}(Y^{+})$$

In general if we have a wall

$$W_\alpha = \{ \zeta \in \mathbb{R}^I \mid \zeta \cdot \alpha = 0 \}$$

$$\alpha \in \mathbb{Z}^I_{\geq 0}$$

$$\subset \mathbb{R}^I$$



How does the generating function behave?

$$Z^{\zeta^-}_{A,P_i}(\vec{q}) = Z^{\zeta^+}_{A,P_i}(\vec{q}) \cdot \left(Z^{\zeta}_{A,\alpha}(\vec{q}) \right)^{\alpha_i}$$

$$\alpha_i = \langle P_i, \alpha \rangle$$

$$Z^{\zeta}_{A,\alpha}(\vec{q}) = \lim_{p^{1/2} \to -1} A(x)\, A^{-1}(p x)$$

$$x = q^{\alpha}$$

$A(x)$ = generating function of counting invts of s.s. A-modules. with

<u>dim</u>

$$\lim_{p^{1/2} \to -1} A(x) A^{-1}(px) \qquad x = q^{\alpha}$$

$A(x)$ = generating function of counting invts of semistable A-modules with $\underline{\dim} = m \cdot \alpha$

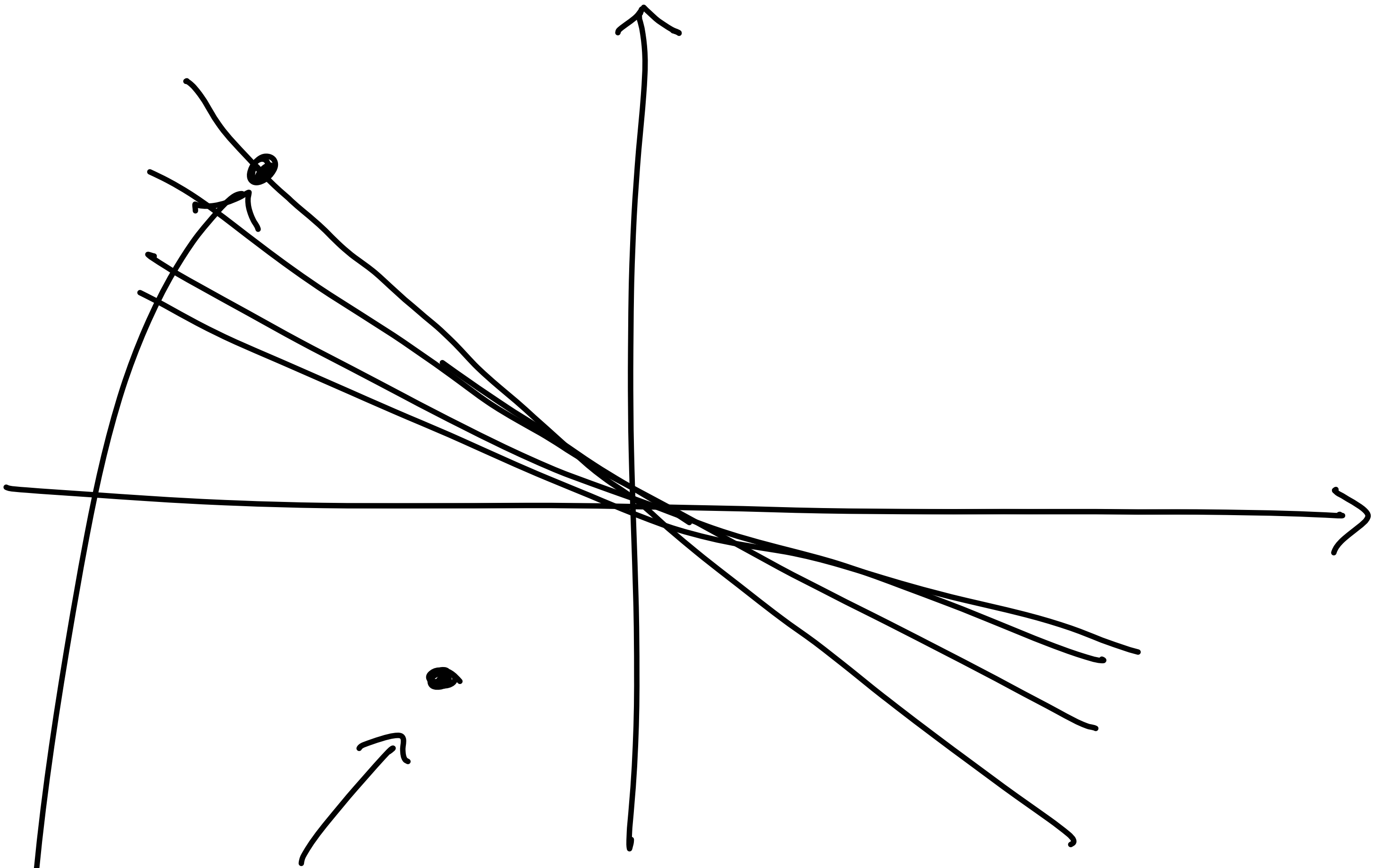
$m \in \mathbb{Z}_{>0}$

N.B. A-modules w/out framing.

Difficult to compute in general.

(Rmk: This is just the semiprimitive w.c.f. in physics. The unframed objects are the halo particles.)

$$\prod (1 - x^n)^{n \Omega(n)}$$



Conifold example

Moduli space $\{ P_0 \to V \}$

cyclic objects in $\underline{\text{mod}}\, A$

$$\{ \overset{P_0}{\overset{\shortparallel}{\mathcal{O}_Y}} \to F \} \text{ in } \operatorname{Coh} Y$$

$$\mathcal{M}^{\zeta_m}_{A,P_0} = \{ (P_0 \twoheadrightarrow V) \text{ in } \text{mod}\, A\cancel{\otimes}\zeta^{-m} \}$$

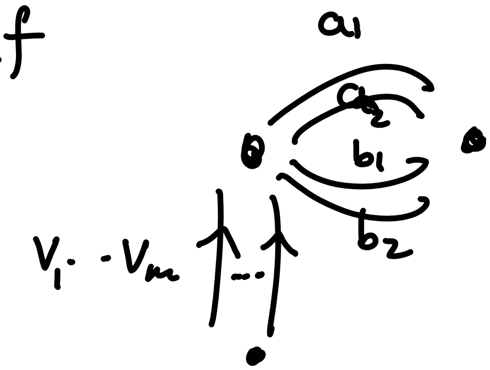
$$\mathcal{L} = \pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$$

$$\begin{array}{c} D^b(\mathrm{mod} A) \\ \wr\!\wr \\ D^b(\mathrm{Coh}\, Y) \circlearrowleft \end{array}$$

so $\otimes \mathcal{L}^{-m}$ makes sense.

- $\left(P_0 \xrightarrow{s} \begin{array}{c} V \\ \cap \\ \mathrm{mod}_f A \otimes \mathcal{L}^{-m} \end{array} \right) \Longleftrightarrow$ suff. large m: $V \otimes \mathcal{L}^{-m} \in \mathrm{mod} A$

moduli of cyclic modules



different t-structures related by mutation.

Local $\mathbb{P}^2$:

- Chamber structure is different ...
- we don't know $A(x)$ don't know $\Omega(n)$'s .....