Jan. 16, 2009

B. Pioline, D-Instantons + Twistors
(Alexandrov, Saueressig, Vandoren)

Try to extend discussion of GMN to string theory, gravity
(Comments on HM moduli spaces)

To linear order near uncorrected semiflat metric.

HK $\longrightarrow$ QK

- <u>complex symplectic</u> quaternionic Kähler.
- symplectomorphisms $\longrightarrow$ contact transformations
- Growth of Ω, won't have anything to say
- NS5 instantons: genuine contact trans ~~not~~ not symplectic

already there in motivic KS
tmns

- Automorphy — $\exists$ $SL(2,\mathbb{Z})$ action.

$\mathcal{N}=4$ strings : automorphy has been very important.

$SL(2,\mathbb{Z}) \rightsquigarrow Li_2$
from Li_3 underlying GW instantons.

① Landscape of HM moduli spaces
② Twistor methods for QK
③ "Semiflat metric ..."
④ D-instanton effects to first order
⑤ NS 5 brane.

$\mathbb{R}^4$

$\mathrm{II}A/X$

$$SK_k(X)\big|_{2h_{11}} \times QK_{cx}(X)\big|_{4h_{12}+4}$$

$$\| $$

$$c\text{-map}\left(SK_{cx}\big|_{2h_{12}}\right)$$

$+$ D branes / H_{odd}

$+$ NS5 / X

$\mathrm{II}B/Y$ $\quad SK_{cx}(Y)\big|_{2h_{12}} \times \widetilde{QK}_k(Y)$

exact in $(2,2)$

$$\| $$

$$c\text{-map}(SK_k(Y))$$

$+$ D branes / H_{even}

$+$ NS5 / Y

$$c\text{-map}(SK_{2n})\big|_{4n+4} \sim \mathbb{R}^+ \times SK_{2n} \times \widetilde{T}_{2n+3}$$

$\tilde{T}_{2n+3}$ = twisted torus

$$\begin{array}{ccc} S_1 & \longrightarrow & \tilde{T}_{2n+3} \\ & & \downarrow \\ & & T_{2n+2} \end{array}$$

Now goto $\mathbb{R}^3 \times S^1$ $\qquad R(S^1) = e^u$

IIA: $QK_{cx}(X)$ uncorrected

IIA / X

$$\begin{array}{ccc} SK_k(X)\big|_{2h_{11}} & \times & QK_{cx}(X)\big|_{4h_{12}+4} \\ \rightsquigarrow & & \| \\ QK_k(X)\big|_{4h_{11}+4} & \times & QK_{cx}(X) \\ \| & & \end{array}$$

c-map$\left(SK_k(X)\big|_{2h_{11}}\right)$ + D / Heun $\times S_1$

+ KK monopoles

So computing $QK_k(X)$ captures BPS spectrum

IB/Y

$$\begin{tikzcd} SK_{cx}(Y)\big|_{2h_{12}} \times \widetilde{QK}_k(Y) \arrow[d] \arrow[d, phantom, shift left=20ex, "||"] \\ \widetilde{QK}_{cx}(Y)\big|_{4h_{12}+4} \times \widetilde{QK}_k(Y) \end{tikzcd}$$

<u>on S^1</u>

C-map(SK_{cx})+D$/H_{odd} \times S_1$

+ KKM

Also $HET^{E_8 \times E_8}/K3 \times T2$

4d $\quad SK(T_2) \times QK(K_3)$ moduli of bundles on a circle

$$\downarrow$$

exactly computed in (4,0) CFT

$$QK(T^3) \times QK(K_3)$$

$$\parallel$$

c-map + corrections.

String dualities give relations

<u>T-duality on S^1</u> $\quad Y = X$, IIA $\leftrightarrow$ IIB

$$QK_k(X) = \widetilde{QK}_k(X)$$

$$QK_{cx}(X) = \widetilde{QK}_{cx}(X)$$

$\therefore$ we can drop the $\sim$

<u>Mirror Symmetry</u> $\quad Y = X^\vee$

$$QK_k(X) = QK_{cx}(X^\vee)$$

Het – type II : (X is K3 fibered)

$$QK_K(X) = QK_K(T^3)$$
$$QK_{cx}(X) = QK(K3)$$

So from (4,0) CFT we can deduce $QK(K3)$.

$$IIA/X = M/X \times S^1$$

$\mathbb{R}^4 \times S^1$: M/X moduli space

$$SR \times QK_{cx}(X)$$

interpret as:

$M/CY \times T^2 \times \mathbb{R}^3 \Rightarrow$ must have an $SL(2,\mathbb{Z})$ isometric action on $QK_K(X)$

$= SL(2,\mathbb{Z})$ of IIB on $\widetilde{QK}_K(T)$

Het/Type II case should have an
$SL(3,\mathbb{Z})$ action.
w/out quantum corrections $SO(3,n;\mathbb{Z})$

Aspects of the geometry of the QK
spaces

II. Twistor methods for QK

1. Hitchin's theorem

$$\mathcal{S}_{4d} \;\; HK \Longleftrightarrow \mathcal{Z}_{\mathcal{S}}$$

$\mathcal{Z}_{\mathcal{S}}$ = complex dim of real dim $4d+2$

$$\begin{array}{c} \mathcal{Z}_{\mathcal{S}} \\ \downarrow p \\ \mathbb{P}_1 \end{array} \qquad \mathcal{Z}_{\mathcal{S}} = \mathbb{P}_1 \times S$$

as a C^∞ mfld

has a family of holomorphic
sections w/ normal bundle $\mathbb{C}^{2d} \otimes \mathcal{O}(1)$

These are the <u>twistor lines</u>

$$\begin{tikzcd} & Z_{\mathcal{S}} \arrow[dl] \arrow[dr] & \\ \mathcal{S} & & {} \end{tikzcd}$$

given $x \in \mathcal{S}$ $\exists$ $\mathbb{CP}^1$ in $Z_{\mathcal{S}}$ S_x
twistor line.

- hol section Ω of $\Lambda^2 T_F^* \mathcal{S}(2)$
- real structure.

Compute twistor space — Local Darboux coordinates:

Patches $\mathcal{U}_i$ of $Z_{\mathcal{S}}$

$\zeta \in \mathbb{P}^1$

$$\Omega^{[i]} = -d\nu^I_{[i]} \wedge d\mu^{[i]}_I \mod d\zeta$$

(ν, μ, ζ) Ω is degenerate

On $\mathcal{U}_i \cap \mathcal{U}_j$

$$\Omega^{[i]} = f_{ij}^2 \, \Omega^{[j]} \quad \text{mod } d\zeta$$

glue together Darboux coord's $\Rightarrow$ must be related by a symplectomorphism

$$S^{[ij]}\left(\nu^I_{[i]}, \mu_{[j]I}, \zeta\right)$$

$$\left.\begin{aligned} \mu_{[i]I} &= f_{ij}^2 \frac{\partial S^{[ij]}}{\partial \nu^{[i]I}} \\ \nu^I_{[j]} &= \frac{\partial S^{[ij]}}{\partial \mu^{[j]}_I} \end{aligned}\right] \quad (*)$$

- on $\mathcal{U}_i \cap \mathcal{U}_j \cap \mathcal{U}_k$ Some complicated cocycle conditions
- mod $S^{[i]}$ in $\mathcal{U}^i$

Trivial symplecto —

$$S^{(ij)} = \nu^I_{[i]} \mu^{[j]}_I$$

Moduli space of solutions to $(*)$

$\sim \mathcal{S}$ by Hitchin thm.

$\nu^I(x^\mu, \zeta)$

Then can get the HK metric from:

$$\Omega = \omega_{2,0} + \zeta\omega_{11} + \zeta^2\omega_{0,2}$$

2. For QK manifolds: we use the Swann construction (superconformal quotient)

$$\begin{array}{ccc} \mathcal{S}_{4d+4} & \leftarrow & \mathbb{C}^2/\mathbb{Z}_2 \\ \downarrow & & \\ \mathcal{M}_{4d} & & \end{array} \qquad \mathcal{M}_{4d}: \text{QK}$$

$\mathcal{S}_{4d+4}$ = Hyperkähler cone

homothetic Killing field

isometric $SU(2)$ action rotating the 3 quaternionic structures.

Cone $\Longrightarrow$ $S^{[ij]}$ are homogeneous of degree 1 (sections of $\mathcal{O}(2)$)

wrt v^I degree $2n$

μ_I $2-2n$.

<u>Shortcut: LeBrun's theorem:</u>

Rule of thumb

Swann

$\mathcal{M} \longleftrightarrow Z_{\mathcal{M}} = \mathcal{S}/\mathbb{C}^*$

has complex contact structure

holo 1-form up to ----

homogeneous symplectic form is equivalent to contact structure one dimension lower

$$\mathcal{X} = \nu^I d\mu_I \text{ is contact structure on } \mathcal{S}/\mathbb{C}^*$$

$$\mathrm{Holon}(\mathcal{M}) = SU(2) \times Sp(n) \subset SO(4n)$$

$$\vec{p}$$

$$\mathcal{Z}_{\mathcal{M}}: \quad \underset{z}{\mathbb{P}_1} \longrightarrow \mathcal{Z}$$

$$\downarrow$$

$$\mathcal{M}$$

$$Dz = dz + (p_1 + ip_2) - p_3 z - (p_1 - ip_2) z^2$$

(1,0) form.

$\exists\ \phi \quad \mathcal{X}^{[i]} = 2 e^{\phi_i} \dfrac{Dz}{z}$ is holomorphic 1-form on $\mathcal{Z}_{\mathcal{M}}$

Φ_i is holomorphic on $\mathbb{P}^1$

Φ_i – "contact potential"

Same story applies

$$X^{[i]} = d\alpha + \xi^\Lambda d\tilde{\xi}_\Lambda$$

$\mathcal{M}_{4n}$ $\qquad$ $Z_{\mathcal{M}}\big|_{4n+2}$

$I = 1, \dots, n+1$

$\Lambda = 1, \dots n$

What used to be symplectomorphisms now become contact trans

$$(\xi_{[i]}, \tilde{\xi}_{[i]}, \alpha_{[i]})$$

$$\xi^{\Lambda}_{[j]} = f_{ij}^{-2} \frac{\partial S^{(ij]}}{\partial \xi^{[j]}_{\Lambda}}$$

$$\tilde{\xi}^{[i]}_{\Lambda} = \frac{\partial S}{\partial \xi^{\Lambda}_{[i]}}$$

$$\alpha^{[i]} = S^{[ij]} - \xi^{\Lambda[i]} \frac{\partial}{\partial \xi^{\Lambda[i]}} S^{(ij]}$$

$$e^{\phi_i} = \hat{f}^2_{ij} e^{\phi_j}$$

$$\hat{f}^2_{ij} = \partial_{\alpha(j)} S^{(ij]}$$

Symplectomorphisms would have

$$S^{[ij]} = \alpha_{[j]} + f(\xi, \tilde{\xi})$$

III. "Semiflat Metric"

c-map (SK)

suppose SK defined by prepotential

$F(X^\Lambda)$ $\quad \mathcal{M}_{cx}$: $\quad X^\Lambda = \int_{\gamma^\Lambda} \Omega$

$$F_\Lambda = \int_{\gamma_\Lambda} \Omega$$

~~IIB~~ (CY)

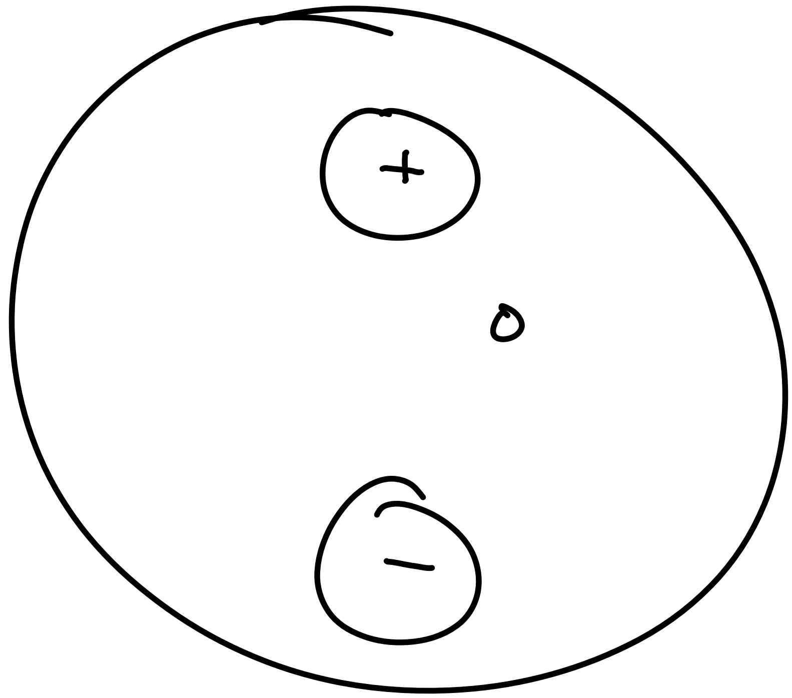
$\cdot\ \mathcal{M}_K$

$$F = \frac{-k_{abc} X^a X^b X^c}{X^0} + \chi \frac{\zeta(3)(X^0)^2}{2(2\pi i)^3}$$

$$- \frac{(X^0)^2}{(2\pi i)^3} \sum n^{(0)}_{k_a} \mathrm{Li}_3\left(e^{2\pi i k_a X^a / X^0}\right)$$

c-map: $\mathbb{R}^+ \times SK \times \widetilde{T}^{2n+3}$

Transition functions are simple



Transition functions

$$S^{[0+]} = \alpha + \xi\tilde{\xi} - \frac{i}{2} F(\xi^\Lambda)$$

$$S^{[0-]} = - \frac{i}{2} \bar{F}(\xi^\Lambda)$$

This is the tree level answer.
Limit at large volume

$IIA/S^1 \quad R \to \infty \quad$ CY has finite volume

IIB $\quad g_{str}^4 \longrightarrow 0$

CY moduli fixed

1 - loop correction:

allow for log singularities
at north and south pole
NOT present in field theory
but are present in string theory.

$$X_\gamma = q_\Lambda \xi^\Lambda - p^\Lambda \tilde{\xi}_\Lambda$$

Twistor lines are simple:

$$\Theta_\gamma \qquad \tilde{T}^{2n+3} \qquad \xi^\Lambda \; \tilde{\xi}_\Lambda \; \sigma$$

$$\Theta_\gamma = q_\Lambda \xi^\Lambda - p^\Lambda \tilde{\xi}_\Lambda$$

$$X_\gamma = R\frac{Z_\gamma}{z} + \theta_\gamma - R\bar{Z}_\gamma \cdot z$$

$$\alpha = \frac{RW}{z} + \sigma - R\bar{W}z \pm \frac{i\chi}{24\pi}\log z$$

$z = \zeta$ in GMN

$$W = X^\Lambda \zeta_\Lambda - F_\Lambda \tilde{\zeta}^\Lambda$$

These are twistor lines on the equator. Don't understand $\pm\frac{i\chi}{24\pi}\log z$ term very well.

IV. Instanton corrections

Expect contributions from branes

But $SL(2,\mathbb{Z})$ constraint is very powerful.

In the absence of GW corrections and $\chi = 0$ then $SL(2,\mathbb{R})$ acts isometrically on the moduli space.

Can compute metric + Killing vectors

Produce global $\mathcal{O}(2)$ sections on $\mathcal{Z}_{\mathcal{M}}$ satisfying $sl(2)$ algebra under contact Poisson bracket.

$\Longleftrightarrow$ Killing vector by moment map construction

$$\Lambda = (0, a)$$

$$\xi^0 \to \frac{a\xi^0 + b}{c\xi^0 + d}, \quad \xi^a \to \frac{\xi^a}{c\xi^0 + d}$$

but action on magnetic coordinate is much more involved:

$$\tilde{\xi}_a \to \tilde{\xi}_a + \frac{i\,c}{4(c\xi_0+d)} K_{abc}\,\xi^b \xi^c$$

$$\begin{pmatrix} \tilde{\xi}_0 \\ \alpha \end{pmatrix} \to \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \xi_0 \\ \alpha \end{pmatrix} + \text{-----}$$

descends to action on the QK mfld.

To determine Kähler potential on

Z = Kähler Einstein with Lorentzian signature

K_{Z_M} is determined by contact potential ϕ

$$e^{\phi} \to \frac{e^{\phi}}{|c\tau + d|} \Rightarrow K_Z \text{ transfs by Kähler transf.}$$

$$\tau = \zeta^0 + 2iR$$

= 10D axiodilaton

@ Rocek et. al. : Can restore an $SL(2,\mathbb{Z})$ action of $SL(2,\mathbb{R})$ by including D instantons and D1 instantons. Doesn't break ½ isoms

(there will also be D3 and D5 instantons)

$\Rightarrow$ ξ will continue to be good coord's.

<u>Rough idea:</u> $\sum n \, Li_3(Y)$

$$Li_3(Y) = \sum \frac{y^n}{n^3}$$

Covariantize that by $\displaystyle\sum \frac{y^n}{(n+m\xi^0)^3}$

Can compute twistor space description

$\longrightarrow$ do a Poisson resumation
$n \to k_0$ and get

$$\sum_{k_0} Li_2(\tilde{y}_{k_0})$$

e^{ϕ} = perturbative piece

$$+ \sum_{q_\Lambda} n^{(0)}_{q_\Lambda} \sum_{m=1}^{\infty} \frac{|q_\Lambda Z^\Lambda|}{m} \cos(2\pi m q_\Lambda \zeta^\Lambda)$$

$$\cdot K_1(2\pi m |q_\Lambda Z^\Lambda| \tau_2)$$

$$Z^\Lambda = \begin{pmatrix} 1 \\ z^a \end{pmatrix} \qquad z^a = X^a / X^0$$

$$\gamma^\Lambda q_\Lambda \in \underset{D1}{H_2(X, \mathbb{Z})} \oplus \underset{D(-1)}{H_0(X, \mathbb{Z})}$$

Take mirror dual

$\in$ Lagrangian sublattice of $H_3(X^\vee, \mathbb{Z})$

Covariantize under symplectic invariance

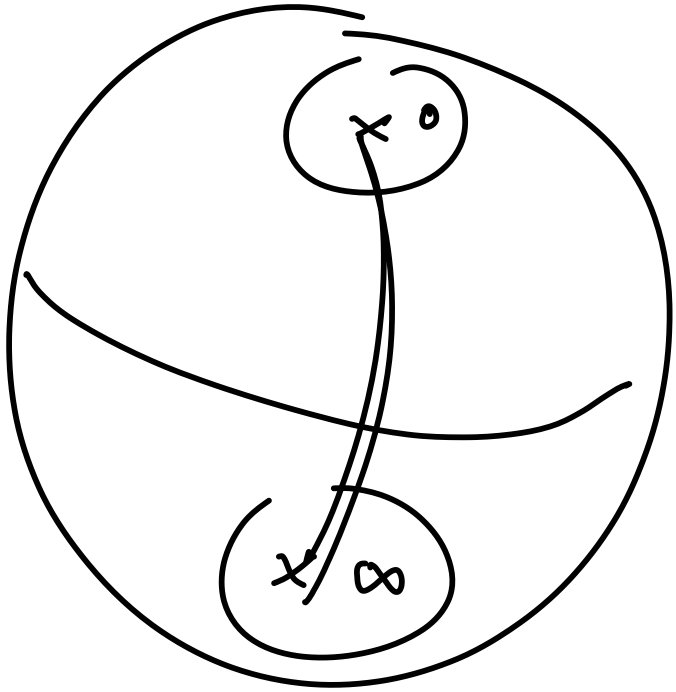
$\sum_q \to \sum_{p,q}$ $|q_\Lambda z^\Lambda| \longrightarrow |Z_\gamma|$ etc.

but this is only correct to leading order away from leading mfld where $QK|_{p^\Lambda=0}$.

$$Z_\gamma = q_\Lambda z^\Lambda - p^\Lambda F_\Lambda$$

In general can't just superimpose terms in Kähler potential.

Extract the $S^{[ij]}$



as in GMN

all BPS rays collapsed to one line

Known symplectic transformation from north pole to the equator.

$$\sum_{\gamma=(p^\Lambda, q_\Lambda)} \underbrace{n_\gamma}_{d} \sum_{m=1}^{\infty} \frac{|Z_\gamma|}{m} \cos(2\pi m \theta_\gamma)$$

Extra contributions from NS 5 branes

Weighted by $e^{ik\sigma}$ $\quad k \in \mathbb{Z} \setminus \{0\}$

$\sigma =$ NS axion

On the twistor space must have some $e^{ik\alpha}$

$$\alpha = \sigma + R\left(\frac{w}{z} - \bar{w} z\right) + \chi \log z + \cdots$$

algebra of translations on twisted torus is a Heisenberg algebra

$$\chi = d\alpha + \xi \, d\tilde{\xi}$$

$$P^\Lambda = \frac{\partial}{\partial \tilde{\xi}_\Lambda} \qquad k = \frac{\partial}{\partial \alpha}$$

$$Q_\Lambda = \frac{\partial}{\partial \xi^\Lambda} - \tilde{\xi}^\Lambda \frac{\partial}{\partial \alpha}$$

translations on magnetic torus

$$[P^\Lambda, Q_\Sigma] = -k\,\delta^\Lambda{}_\Sigma$$

So $$X_\gamma = \exp\left(i\, P^\Lambda q_\Lambda - Q_\Lambda p^\Lambda\right)$$

$$X_\gamma X_{\gamma'} = X_{\gamma'} X_\gamma\, q^{\langle\gamma,\gamma'\rangle}$$

$$q = e^{iK}$$

So any function we can decompose in Fourier modes : Better ireps of Heisenberg group

$$S^{[ij]} = \sum_\gamma n_\gamma \exp(W_\gamma)$$

$$+ \sum_{K=1}^{\infty} \sum_{n^\Lambda \in \frac{\Gamma_m}{|K|\Gamma_m}} \sum_F \theta_{K,n,F}^{(\xi,\tilde{\xi},\alpha)}$$

$$\Gamma = \Gamma_e + \Gamma_m$$

$$W = p^\Lambda \tilde{\xi}_\Lambda - q_\Lambda \xi^\Lambda$$

θ series are inv under Heisenberg group

$$\Theta_{k,n,F} = \sum_{l^\Lambda \in \Gamma_m + \frac{n^\Lambda}{|k|}} F(\xi^\Lambda + l^\Lambda) \exp\left(2\pi i\left(\xi_\Lambda l^\Lambda + k\alpha\right)\right)$$

Theta series.

$\Theta \sim$ new invts of CY.

what F's to plug in.

e.g. $F = Z_{top}(\xi^\Lambda)$

If you could restrict to this state .... would be forcing