D-instantons and Twistors

Workshop on derived geometry
Stony Brook, 16/11/09

- Goal: extend GMN's analysis of moduli space of $\mathcal{N}=2$ gauge theories on $\mathbb{R}^3 \times S^1$ to string IIA/IIB on $\mathbb{R}^3 \times S^1 \times CY$
- More generally, analyze the moduli space of hypermultiplets in $\mathcal{N}=2$ string and M-theory compactifications
- As Greg briefly discussed at the end of his talk, some aspects of this gen are straightforward, but new conceptual issues arise / ingredients come into play:
  - HK geometry $\longrightarrow$ QK geometry
    Symplectomorphisms $\longrightarrow$ contact transformations
  - Growth of BH degeneracies / generalized DT invariants
  - NS5-brane instantons / KKM require further generalizing DT gen (perhaps motivic DT ?)
  - Automorphicity :
    - $\mathrm{IIA}/CY \times S_1 = M/CY \times T^2$, with $T^2 \leftarrow Sl(2,\mathbb{Z})$
    $\qquad \|$
    $\mathrm{IIB}/CY \times S_1$
    $\qquad \|$
    $\mathrm{Het}/K3 \times T^3$, with $K3 \leftarrow SO(4,n,\mathbb{Z})$ and $T^3 \leftarrow Sl(3,\mathbb{Z})$, $SO(3,n,\mathbb{Z}), \ldots$
    - important for $\mathcal{N}=4$ dyons
- We shall content ourselves with analyzing corrections to the weak coupling ("semi-flat") metric to leading order.

$$\underset{GW}{Li_3} \xrightarrow{Sl(2,\mathbb{Z})} \underset{\text{generalized DT}}{Li_2}$$

Outline

1. Physics motivation. landscape of HM moduli spaces
2. Twistor methods for QK manifolds
3. The "semi-flat" metric
4. Instanton corrections, to leading order

References

Robles Llana Rocek Saueressig Theis Vandoren hep-th/0612027

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(Neitzke BP Vandoren hep-th/0701214)

Aspinwall's lectures hep-th/0001001

⋮

I. Landscape of HM moduli spaces

$M/X \times \mathbb{R}^5$

$SR|_{h_{11}-1} \times QK_{cx}(X)|_{4h_{12}+4}$

$\|$

$\underline{IIA}/X \times \mathbb{R}^4$

$SK_K(X)|_{2h_{11}} \times QK_{cx}(X)|_{4h_{12}+4}$

Quantum cohom. / ws. instantons

$QK_{cx}(X) = $ c-map$(SK_{cx}(X))$ + Dbranes$/H_{odd}(X)$ + NS5$/X$

$\underline{IIB}/Y \times \mathbb{R}^4$

$SK_{cx}(Y)|_{2h_{12}} \times \widetilde{QK}_K(Y)|_{4h_{11}+4}$

exact in (2,2) SCFT

Lag Cone in $H_3(Y,\mathbb{C})/\mathbb{C}^\times$

$\widetilde{QK}_K(Y) = $ c-map$(SK_K(Y))$ + D-branes$/H_{ev}$ + NS5$/Y$

$\mathrm{Het}^{E_8 \times E_8}/K3 \times T^2 \times \mathbb{R}^4$

$SK(T^2) \times QK(K3)$

$\uparrow$ exact in (0,4) SCFT

$I/K3 \times T^2 \times \mathbb{R}^4$

...

$\underline{IIA}/X \times \mathbb{R}^3 \times S^1$

$QK_K(X)|_{4h_{11}+4} \times QK_{cx}(X)$

$QK_K(X) = $ c-map$(SK_K(X))$ + Dbranes$/H_{ev} \times S_1$ + KK monopoles

$\underline{IIB}/Y \times \mathbb{R}^3 \times S_1$

$\widetilde{QK}_{cx}(Y)|_{4h_{12}+4} \times \widetilde{QK}_K(\cancel{X})$

$\widetilde{QK}_{cx}(Y) = $ c-map$(SK_{cx}(Y))$ + D-branes$/H_{odd} \times S_1$ + KK monopoles

$\mathrm{Het}/K3 \times T^2 \times \mathbb{R}^3 \times S^1$

$QK(T^3) \times QK(K3)$

$QK(T^3) = $ c-map$(SK(T^2))$ + BH$/S_1$ + KKM

NB

- c-map(SK_{2n}) produces a $QK|_{4n+4}$ manifold $\mathbb{R}^+ \times SK_{2n} \times \tilde{T}_{2n+3}$, where $\tilde{T}_{2n+3} = $ "twisted torus" $S_1 \to \tilde{T}_{2n+3} \to T_{2n+2}$
- D-branes on S_1 are black holes in 4D!

T-duality along S_1 $\Rightarrow$ $X = Y$

$$QK_K(X) = \widetilde{QK}_K(X)$$
$$QK_{cx}(X) = \widetilde{QK}_{cx}(X)$$

: we can drop the tilde!

$QK_K(X)|_{4h_{11}+4}$

$QK_{cx}(X)|_{4h_{12}+4}$

Mirror sym $\Rightarrow$ $X^\vee = Y$

$$QK_K(X) = QK_{cx}(X^\vee)$$
$$QK_{cx}(X) = QK_K(X^\vee)$$

String-string duality $\Rightarrow$ (if X is K3 fibered)

$$QK_K(X) = QK(T^2)$$
$$QK_{cx}(X) = QK(K3)$$

for appropriate choices of $E_8 \times E_8$ bundle over $K_3 \times T_2$

Het / $K3 \times T_2$ is the place of choice to compute $QK(K3)$
(SCFT (4,0) on K3)

M-theory lift: $\quad IIA / X \times S^1 = M / X \times T^2$

so $\quad QK_K(X)$ must admit an isometric $Sl(2,\mathbb{Z})$ action!

(Similarly $\widetilde{QK}_K(Y)$ must admit an iso $Sl(2,\mathbb{Z})$ action – the same one)

$QK(T^3)$ must also admit an $Sl(3,\mathbb{Z})$ isometric action
or perhaps even $SO(3,n,\mathbb{Z})$

$QK(K3)$ probably has an $SO(4,n',\mathbb{Z})$ action ...

$\Rightarrow$ Automorphy is a strong constraint on $QK_K(X)$ and $QK_{cx}(X)$

II Twistor methods for QK manifolds

2.1. Recall Hitchin's theorem:

$$\mathrm{HK}_{4d} : \mathcal{S} \longleftrightarrow Z_{\mathcal{S}}$$

$$\begin{array}{c} Z_{\mathcal{S}} \\ p \downarrow \\ \mathbb{P}^1 \end{array}$$

- cx manifold of dim $4n+2$
- with holomorphic section Ω of $\Lambda^2 T^*_F(2)$ defining a symplectic form on each fiber
- family of hol. sections ("twistor lines") with normal bundle $\sim \mathbb{C}^{2d} \otimes \mathcal{O}(1)$
- real structure

This means that $Z_{\mathcal{S}}$ can be constructed by patching together $\mathcal{U}^{[i]}$ flat $\mathbb{C}^{2d}$ with local Darboux coordinates $\nu^I_{[i]}, \mu_{I[i]}$:

$$\Omega^{[i]} = d\mu_I^{[i]} \wedge d\nu^I_{[i]}$$

$$\Omega^{[i]} = f_{ij}^2 \, \Omega^{[j]} \mod d\xi \qquad f_{ij} = \text{transition fct of } \mathcal{O}(1)$$

On $\mathcal{U}_i \cap \mathcal{U}_j$: symplectomorphism generated by hol functions

$$S^{[ij]}(\nu_{[i]}, \mu_{[j]}, \xi)$$

$$\mu^I_{[i]} = f_{ij}^{+2} \frac{\partial S^{[ij]}}{\partial \nu^I_{[i]}} , \qquad \nu_I^{[j]} = \frac{\partial S^{[ij]}}{\partial \mu^{[j]}} \qquad (*)$$

with obvious compatibility conditions on $\mathcal{U}_i \cap \mathcal{U}_j$, modulo local symplectomorphisms $S^{[i]}$ on $\mathcal{U}_i$.

($\underline{\text{Rk}}$ for infinitesimal diffeo: $S^{[ij]} = f_{ij}^{-2n} \nu^I_{[i]} \mu_I^{[j]} - \tilde{H}(\nu_{[i]}, \mu^{[j]}, \xi)$
$\tilde{H}^{[ij]}$ defines a section of $H^1(Z, \mathcal{O}(2))$)

- The HK metric can be obtained by "parametrizing the real twistor lines", ie solutions of (*) regular in each patch:

$$\nu^{I}_{[i]}(\xi ; z_i) \,,\quad \mu^{[i]}_{I}(\xi ; z_i)$$

and expanding $\Omega^{[i]}$, say for $i=0$ (north pole)

$$\Omega^{[0]}_{(\xi)} = \omega_{2,0} + \xi \underbrace{\omega_{1,1}}_{\text{Kähler form}} \oplus \xi^2 \omega_{0,2}$$

- Trihol isometries $\leftrightarrow$ global $\mathcal{O}(2)$-sections

2.2. This works for QK manifolds as well, using Swann's construction

$$\begin{array}{ccccc} QK_{4d} & \longleftrightarrow & HKC_{4d+4} & \longleftrightarrow & Z_{\mathcal{S}} \\ \mathcal{M} & & \mathcal{S} & & \end{array}$$

"Swann bundle" — Twistor space with extra homogeneity constraint

- Recall that $\mathcal{M}$ has reduced hol. $SU(2)\times USp(d) \subset SO(4d)$

$$T\mathcal{M} = E_{2n} \otimes H_2 \qquad \text{(Salamon)}$$

$\mathcal{S}$ is the total space of $(H \setminus \{0\})/\mathbb{Z}_2$ bundle over $\mathcal{S}$.

- $\mathcal{S}$ admits a canonical HK^* metric with homothetic vector and isometric action of $SU(2)$, rotating the 3 cx structures.
- Any isometry of $\mathcal{M}$ lifts (uniquely) to a triholomorphic isom of $\mathcal{S}$
- The symplectic structure $\Omega^{[i]}$ is required to be homogeneous. Equiv, the transition functions $S^{[ij]}$ must be local $\mathcal{O}(2)$ sections, while $\nu_{[i]}, \mu_{[i]}$ are local $\mathcal{O}(2n)$, $\mathcal{O}(2-2n)$ sections

- Crucial and trivial fact:

$$\text{Homogeneous symplectic structure on } \mathbb{C}^{2d+2} \;=\; \text{Contact structure on } \mathbb{C}^{2d+1}$$

Indeed, $\mathcal{Z}_{\mathcal{M}} = \mathcal{S}/\mathbb{C}^\times$ is a $\mathbb{P}^1$ bundle over $\mathcal{M}$, with a canonical complex structure and a complex contact structure: given by the kernel of the $(1,0)$ form

$$Dz = dz + p_+ + p_3 z - p_- z^2$$

- LeBrun's theorem gives a 1-1 correspondence between QK manifolds and (Fano) complex contact manifolds $\mathcal{Z}$ with a real structure.

[Note: $\displaystyle ds^2_{\mathcal{Z}} = \frac{|Dz|^2}{(1+z\bar{z})^2} + \frac{\nu}{4} ds^2_{\mathcal{M}}$ gives a Kähler-Einstein metric on $\mathcal{M}$]

- Just as in HK case, a complex manifold can be obtained by gluing together open sets $\mathcal{U}_i$ in $\mathbb{C}^{2d+1}$ using contact transf.

On $\mathcal{U}_i$: Darboux coordinates $\xi^\Lambda_{[i]}, \tilde{\xi}^{[i]}_\Lambda, \alpha^{[i]}$

$$\mathcal{X}^{[i]} = d\alpha + \xi^\Lambda d\tilde{\xi}_\Lambda = 2 e^{\Phi_i} \frac{Dz}{z}$$

where Φ_i is holomorphic along the fibers.

Contact transformations are generated by $S^{[ij]}(\xi^\Lambda_{[i]}, \tilde{\xi}^{[j]}_\Lambda, \alpha^{[j]})$

$$\begin{cases} \xi^\Lambda_{[j]} = f_{ij}^{-2} \dfrac{\partial}{\partial \tilde{\xi}^{[j]}_\Lambda} S^{[ij]} \\ \tilde{\xi}^{\Lambda}_{[i]} = \dfrac{\partial}{\partial \xi^\Lambda_{[i]}} S^{[ij]} \\ \alpha^{[i]} = S^{[ij]} - \xi^\Lambda_{[i]} \dfrac{\partial}{\partial \xi^\Lambda_{[i]}} S^{[ij]} \end{cases}$$

where $f_{ij}^2 = \partial_{\alpha^{[j]}} S^{[ij]}$

The "contact potential" Φ_i satisfies $e^{\Phi_i} = f_{ij}^2 e^{\Phi_j}$

From the twistor lines $\xi^\Lambda(x^\mu, z)$, $\tilde{\xi}_\Lambda(x^\mu, z)$, $\alpha(x^\mu, z)$

and contact potential $\phi(x^\mu, z)$ one easily extracts

the $SU(2)$ connection $\vec{p}$, the quaternionic forms $\vec{\omega} = \frac{2}{\nu}(d\vec{p} + \vec{p} \wedge \vec{p})$

and the QK metric

In general: hard to find the twistor lines. ↗ "toric QK"

It is easy however when $\mathcal{M}$ admits $d+1$ ~~[illegible]~~ commuting isometries:

any isometry lifts to a holomorphic isometry of $\mathcal{Z}$, and produces a global $O(2)$ section ξ^Λ

$$\Rightarrow \qquad S^{[ij]} = \alpha^{[j]} + \xi^{\Lambda[i]} \tilde{\xi}_\Lambda^{[j]} - H^{[ij]}(\xi^\Lambda_{[i]})$$

$$\begin{cases} \xi^\Lambda = \dfrac{Y^\Lambda}{z} + \xi^\Lambda - \bar{Y}^\Lambda z \\ \tilde{\xi}_\Lambda^{[i]} = \dfrac{i}{2} \xi_\Lambda + \dfrac{1}{2} \sum \oint \dfrac{dz'}{2\pi i z'} \dfrac{z'+z}{z'-z} \partial_{\xi^\Lambda} H^{[0j]}(\xi) \\ \alpha^{[i]} = \dfrac{i}{2}\sigma + \dfrac{1}{2} \sum \oint (\qquad) \left(H^{0j} - \xi \dfrac{\partial}{\partial \xi} H\right)^{[0j]} \end{cases}$$

Rk it is also consistent to allow log divergent terms

$$\tilde{\xi}_\Lambda \to \tilde{\xi}_\Lambda + c_\Lambda \log z$$

$$\alpha \to \alpha + c_\alpha \log z + c_\Lambda (Y^\Lambda / z + \bar{Y}^\Lambda z)$$

Similarly, it is easy to determine the twistor lines when $\mathcal{M}$ is close to a toric QK manifold:

$$S^{[ij]} = \alpha^{[j]} + \xi^{\Lambda[i]} \tilde{\xi}_\Lambda^{[j]} - H^{[ij]}(\xi^\Lambda) - \varepsilon H^{[ij]}_{(1)}(\xi^\Lambda, \tilde{\xi}_\Lambda, \alpha)$$

order by order in ε.

NB $$e^\phi = \frac{1}{4} \sum \oint \frac{dz'}{2\pi i z'} (z'^{-1} Y^\Lambda - z \bar{Y}^\Lambda) \frac{\partial}{\partial \xi^\Lambda} H(\xi(z')) + \frac{1}{2}(c_\Lambda \xi^\Lambda + c_\alpha)$$

III The "semi flat" metric

At zero coupling, or infinite radius,

$QK_{K,cx}(X)$ are given by the "c-map" of $SK_{K,cx}$, resp.

* SK is described by a symplectic section $(X^\Lambda, F_\Lambda)(t_i)$ such that $F_\Lambda = \dfrac{\partial F}{\partial X^\Lambda}$, F is the prepotential, hom degree 2

  - For SK_{cx}: the moduli space of cx structures

$$\begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix} = \int_{\begin{pmatrix} \gamma^\Lambda \\ \gamma_\Lambda \end{pmatrix}} \Omega \qquad \text{periods of } \Omega \text{ on a symplectic basis}$$

  - For SK_K: the moduli space of complexified Kähler structures,

$$F = -\kappa_{abc}\frac{X^a X^b X^c}{X^0} + \chi\,\frac{\zeta(3)(X^0)^2}{2\cdot(2\pi i)^3} - \frac{(X^0)^2}{(2\pi i)^3}\sum n^{(0)}_{k_a}\, \mathrm{li}_3\!\left(e^{2\pi i k_a X^a/X^0}\right)$$

$k_a \gamma^a \in H_2^+(X)$

Genus 0 Gromov Witten.

* The c-map is a QK manifold of the form

$$\underset{\substack{\uparrow \\ \phi}}{\mathbb{R}^+} \times \underset{2h}{SK} \times \underset{\substack{\uparrow \\ \xi^\Lambda,\, \tilde\xi_\Lambda,\, \sigma}}{T^{2h+3}}$$

Its twistor space is described by transition functions

$$H^{[0+]} = -\frac{i}{2} F(\xi^\Lambda) \qquad \begin{cases} c_\Lambda = 0 \\ c_\alpha = \dfrac{\chi}{96\pi} \quad \text{"1-loop correction"} \end{cases}$$

$$H^{[0-]} = -\frac{i}{2} \bar F(\xi^\Lambda)$$

such that the twistor lines take the form, around u_0,

$$\xi^\Lambda = \xi^\Lambda + \mathcal{R}\left(\frac{Z^\Lambda}{z} - z\bar Z^\Lambda\right)$$

$$\rho_\Lambda = \tilde\xi_\Lambda + \mathcal{R}\left(\frac{F_\Lambda}{z} - z\bar F_\Lambda\right)$$

$$\tilde\alpha = \sigma + R\left(\frac{W}{z} - z\bar W\right) \pm \frac{i\chi}{96\pi}\log z$$

Compare to GMN:

$X_\gamma = \exp\left(i(k_\Lambda \xi^\Lambda - e^\Lambda \tilde\xi_\Lambda)\right)$

if $\gamma = (k_\Lambda \gamma^\Lambda - e^\Lambda \gamma_\Lambda)$

Where

$$e_\Lambda = -2i\,\tilde{\xi}_\Lambda \qquad R = \frac{\tau_2}{2}$$
$$\tilde{\alpha} = 4i\alpha + 2i\,\tilde{\xi}_\Lambda \xi^\Lambda$$
$$W = F_\Lambda \xi^\Lambda - z^\Lambda \tilde{\xi}_\Lambda$$
$$K = \frac{-2}{2i}\left(\bar{z}F - z\bar{F}\right)$$

and GW instantons

* In the absence of 1-loop correction, $QK_K(X)$ admits an isometric $Sl(2,\mathbb{R})$ action.

The simply way to see it is to produce global $\mathcal{O}(2)$ sections satisfying the $Sl(2)$ algebra under Poisson bracket:

$$\mu^+ = -\tilde{\xi}_0^{[0]}$$
$$\mu^0 = \alpha^{[0]} - \xi^0 \tilde{\xi}_0^{[0]}$$
$$\mu_- = \alpha^{[0]} - \frac{i}{12} K_{abc}\, \xi^a \xi^b \xi^c$$

are manifestly regular in $\mathcal{U}_0$. Regularity in $\mathcal{U}_\pm$ can be checked using contact-transformed vars.

$$\xi^0 \to \frac{a\xi^0 + b}{c\xi^0 + d} \qquad \xi^a \to \frac{\xi^a}{c\xi^0 + d}$$

$$\tilde{\xi}_a \to \tilde{\xi}_a + \frac{ic}{4(c\xi_0 + d)} K_{abc}\, \xi^b \xi^c$$

$$\begin{pmatrix} \tilde{\xi}_0 \\ \alpha \end{pmatrix} \to \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \begin{pmatrix} \tilde{\xi}_0 \\ \alpha \end{pmatrix} + \frac{i}{12} K_{abc}\, \xi^a \xi^b \xi^c \begin{pmatrix} c^2/(c\xi_0 + d) \\ -(c^2(a\xi_0 + b) + 2c)/(c\xi_0 + d)^2 \end{pmatrix}$$

in such a way that $X \to X/(c\xi^0 + d)$

* This descends to the standard action of $Sl(2,\mathbb{R})$ on

$$\phi,\ z_i,\ \underbrace{\xi^\Lambda, \tilde{\xi}_\Lambda}_{RR},\ \underbrace{\sigma}_{NS\ axion} \quad ; \text{ moreover } \begin{cases} e^\phi \to e^\phi / |c\tau + d| \\ K_z \to K_z - \log|c\xi_0 + d| \end{cases}$$

IV Instanton corrections

Strategy: . use $Sl(2,\mathbb{Z})$ to map WS. instantons (GW invariants) to $D(-1)$ and $D(1)$ instantons (torsion sheaves, support on curves)

. Mirror sym $\to$ D2 branes on A-cycles (SLAG)

. Symplectic invariance A-cycles $\leftrightarrow$ B-cycles

. Mirror sym $\to$ $D(-1), D1, D3, D5$ (general sheaves)

First step: (Robles Llana et al)

$$b^a + i t^a = z^a$$

$$e^{\Phi} = \frac{\tau_2^2}{2} V(t^a) - \frac{\chi \zeta(3)}{8(2\pi)^3}\tau_2^2 - \frac{\chi}{192\pi}$$

$$+ \frac{\tau_2^2}{4(2\pi)^3} \sum_{k_a \gamma^a \in H_2^+} \underbrace{n^{(0)}_{q_a} \mathrm{Re}\left[Li_3\left(e^{2\pi i q_a z^a}\right) + 2\pi q_a t^a \, Li_2\left(e^{2\pi i q_a z^a}\right)\right]}_{e^{\Phi_{WS}}}$$

$$\downarrow$$

$$\frac{\tau_2^2}{2} V(t^a) + \frac{\sqrt{\tau_2}}{8(2\pi)^3} \sum_{k^a} n^{(0)}_{q_a} \sum_{m,n}{}' \frac{\tau_2^{3/2}}{|m\tau+n|^3}\left(1 + 2\pi |m\tau+n| q_a t^a\right) e^{-S_{m,n,q_a}}$$

$$S_{m,n,k_a} = 2\pi q_a \, |m\tau+n| \, t^a - 2\pi i q_a (m c^a + n b^a)$$

$\downarrow$ Poisson resum on $n \to k_0$

$$\frac{\tau_2^2}{2} V(t^a) - \frac{\sqrt{\tau_2}}{8(2\pi)^3} \chi \left[\zeta(3)\, \tau_2^{3/2} + \frac{\pi^2}{3} \tau_2^{-1/2} \right] + e^{\Phi_{WS}}$$

$$+ \frac{\tau_2}{8\pi^2} \sum_{k_\Lambda}{}' n^{(0)}_{k_\Lambda} \sum_{m=1}^{\infty} \frac{|k_\Lambda z^\Lambda|}{m} \cos(2\pi k_\Lambda m \zeta^\Lambda)\, K_1(2\pi m |k_\Lambda z^\Lambda| \tau_2)$$

$\Rightarrow$ contributions of $\left\{ \begin{array}{l} D1/D(-1) \\ \text{or} \\ D2/H_3(X,\mathbb{Z}) \end{array} \right.$

$\downarrow$

$$\frac{R^2}{4} K(z,\bar z) + \frac{\chi}{192\pi} + \frac{1}{8\pi^2} \sum_{\gamma} n_\gamma \sum_{m=1}^{\infty} \frac{|W_\gamma|}{m} \cos(2\pi m \theta_\gamma)\, K_1(4\pi m |W_\gamma|)$$

where $\quad W_\gamma = R\,(q_\Lambda z^\Lambda - p^\Lambda F_\Lambda)$

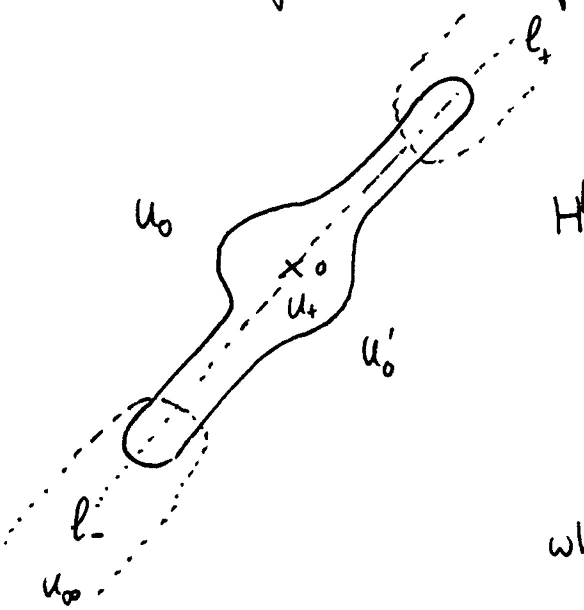
$\qquad\quad \Theta_\gamma = q_\Lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}_\Lambda$

The action

$$4\pi m\,|W_\gamma| \pm 2\pi i m\,\Theta_\gamma$$

fits the expected action of a D2 brane wrapping $q_\Lambda \gamma^\Lambda - p^\Lambda \tilde{\gamma}_\Lambda \in H_3(X)$
or a D(-1)-1-3-5 brane wrapping cycle in H_{even}.

Structure of the twistor space reproducing this contact potential:



$$\ell_\pm = i\mathbb{R}^\pm$$

$$\sum_{m=1}^{\infty} \frac{e^{im\mathcal{Z}_\gamma}}{m^2}\, Ei(-im\mathcal{Z}_\gamma)$$

$$H^{[0+]} = H^{[0'+]} = -\frac{i}{2} F(\xi) + \frac{1}{(2\pi)^3} \sum_{(\gamma)_+}{}' n_\gamma \int_0^{-i\infty} \frac{\mathcal{Z}\,d\mathcal{Z}}{\mathcal{Z}^2 - \mathcal{Z}_\gamma^2}\, Li_2\!\left(e^{-2\pi i \mathcal{Z}}\right)$$

where $\quad \mathcal{Z}_\gamma = q_\Lambda \xi^\Lambda - p^\Lambda \rho_\Lambda$

Sum runs over γ such that $\underline{\underline{Re(W_\gamma) > 0}}$

Across $\ell_\pm$, the symplectomorphism is given (to linear order) by

$$H^{[ij]}_{(1)} = \sum_{(\gamma)\pm} n_\gamma\, Li_2\!\left(e^{\mp 2\pi i (q_\Lambda \xi^\Lambda - p^\Lambda \rho_\Lambda)}\right)$$

which is the linear approximation of $\overset{\curvearrowleft}{\prod_{\pm Re(W_\gamma)}} U_\gamma$

KS symplectomorphisms.

$\rightsquigarrow$ essentially identical to GMN

Using this prescription, we can compute the twistor lines to 1st order

$$\begin{pmatrix} \xi^\Lambda \\ \rho_\Lambda \end{pmatrix} = \begin{pmatrix} \zeta^\Lambda \\ \tilde{\zeta}_\Lambda \end{pmatrix} + R\left[\frac{1}{z} \begin{pmatrix} z^\Lambda \\ F_\Lambda \end{pmatrix} - z \begin{pmatrix} \bar{z}^\Lambda \\ \bar{F}_\Lambda \end{pmatrix} \right]$$

$$+ \frac{1}{16\pi^2} \sum_\gamma n_\gamma \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} I_\gamma^{(1)}$$

$$I_\gamma^{(1)} = \sum_{s=\pm 1} \frac{s}{m} e^{-2\pi i s m \Theta_\gamma} \int_0^\infty \frac{dt}{t} \frac{t - \epsilon_\gamma s i z}{t + \epsilon_\gamma s i z} e^{-2\pi m \epsilon_\gamma \left(\frac{W_\gamma}{t} + t \bar{W}_\gamma \right)}$$

$$\epsilon_\gamma = \mathrm{sgn}(\mathrm{Re}\, W_\gamma)$$

$$\tilde{\alpha} = \dots$$

and obtain the QK metric at this order.

V Discussion :

- $\Omega(\gamma) \sim \exp(\|\gamma\|^2)$ so the $\sum_\gamma$ is unlikely to converge for <u>any</u> value of the coupling, unless some cut off is imposed ...

- NS5 branes must contribute terms proportional to $e^{ik\tilde{\alpha}}$ in the transition functions

$\Longrightarrow$ genuine contact transformations !

$$P^\Lambda = \frac{\partial}{\partial \tilde{\xi}_\Lambda} , \quad Q_\Lambda = \frac{\partial}{\partial \xi^\Lambda} - \tilde{\xi}\, \partial_\alpha , \quad K = \partial_\alpha$$

$$[P^\Lambda, Q_\Sigma] = -K\, \delta^\Lambda{}_\Sigma$$

$$x_\gamma \equiv e^{i(P^\Lambda k_\Lambda - \ell^\Lambda Q_\Lambda)} \qquad x_\gamma\, x_{\gamma'} = x_{\gamma'}\, x_\gamma\, q^{\langle\gamma,\gamma'\rangle}$$

$$q = e^{iK} \qquad \Rightarrow \text{quantum torus !}$$

Define $\gamma = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix}$

$$W_\gamma = R\,(q_\Lambda z^\Lambda - p^\Lambda F_\Lambda) = R\, Z_\gamma$$
$$\theta_\gamma = q_\Lambda \zeta^\Lambda - p^\Lambda \tilde{\zeta}_\Lambda$$

$$X_\gamma = q_\Lambda \xi^\Lambda - p^\Lambda \tilde{\xi}_\Lambda \qquad (\text{the log of GMN's } X_\gamma\,!)$$

$$W = F_\Lambda \xi^\Lambda - z^\Lambda \tilde{\xi}_\Lambda$$

"Twistor lines":

$$\begin{cases} X_\gamma = \theta_\gamma + R\,\dfrac{Z_\gamma}{z} - z\,\bar{Z}_\gamma \\ \tilde{\alpha} = \sigma + R\left(\dfrac{W}{z} - z\,\bar{W}\right) \pm \dfrac{i\chi}{24\pi}\log z \\ e^{\Phi} = \dfrac{R^2}{4}K(z,\bar{z}) \pm \dfrac{\chi}{192\pi} \end{cases}$$

Instanton corrections:

$$X_\gamma = X_\gamma^0 + \frac{1}{16\pi^2}\sum_{\gamma} \gamma \cdot n_\gamma \cdot I_\gamma^{(1)}$$

$$I_\gamma = \sum_{s=\pm 1} \frac{s}{m}\, e^{-2\pi i s m \theta_\gamma} \int_0^\infty \frac{dt}{t}\, \frac{t - \epsilon_\gamma s i z}{t + \epsilon_\gamma s i z}\, e^{-2\pi m \epsilon_\gamma R\left(\frac{Z_\gamma}{t} + t\bar{Z}_\gamma\right)}$$

The space of functions on the twisted complex torus can be decomposed into irreps of the Heisenberg group:

$$H_{(1)}(\xi^\Lambda, \tilde\xi_\Lambda, \alpha) = \sum_{\gamma=(p^\Lambda, q_\Lambda)\in\Gamma} n_\gamma \, e^{i(p^\Lambda \tilde\xi_\Lambda - q_\Lambda \xi^\Lambda)}$$

$$+ \sum_{k=1}^{\infty} \sum_{n^\Lambda \in \Gamma_m / 4|k|\Gamma_m} \sum_F \theta_{k,n,F}(\xi, \tilde\xi, \alpha) \qquad (*)$$

where $\theta_{k,n,F}$ are theta series, $\Gamma = \Gamma_e \oplus \Gamma_m$ is a Lagrangian decomp

$$\theta_{k,n,F} = \sum_{\ell^\Lambda \in \Gamma_m + \frac{n^\Lambda}{4|k|}} F(\xi^\Lambda + \ell^\Lambda) \exp\left(2\pi i k \, \tilde\xi_\Lambda \ell^\Lambda + 2\pi i k \alpha\right)$$

Q: What functions F should enter the sum?

A natural candidate is the top string amplitude $F = \Psi_{top}$!

Suppose F was the only allowed contribution, and $\mathcal{M} = G/H$ was a symmetric space: $(*)$ would be the Fourier-Jacobi decomposition of the minimal representation of G, and the Abelian Fourier coefs would have support on charges with $I_4(\gamma) = 0$.

If this was the case in general, the issue of exponential growth may not arise ...

(The above is <u>very</u> speculative!)