Y. Soibelman, Jan. 15, 2009
Wall Crossing

1.) Reminder on approach to BPS state count

2.) Application to cluster transformations.

Different levels.

Just WCF —

1. graded Lie algebras:

$$\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma \qquad \Gamma = \mathbb{Z}^n$$

Stability data: $Z : \Gamma \to \mathbb{C}$

For every $\gamma \neq 0$ have $a(\gamma) \in \mathfrak{g}_\gamma$

One axiom: For every γ s.t. $a(\gamma) \neq 0$ we should have

$\|\gamma\| \le C|Z(\gamma)|$

Stability condition on (Z, a)

$\longrightarrow$ Hausdorff space $\mathrm{Stab}(\mathfrak{g}) \xrightarrow[\text{local homeo}]{} \{Z\}$

$\exists$ Hausdorff ~~top.~~

$\mathrm{Stab}(\mathfrak{g})$

$\cup$

Walls of MS Can see WCF

Don't need any CY.

$\mathfrak{g}_\gamma = \mathbb{Q} e_\gamma$

$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle\gamma_1,\gamma_2\rangle} \langle\gamma_1,\gamma_2\rangle e_{\gamma_1+\gamma_2}$

$a(\gamma)$ - just #'s. eventually lead to $\Omega(\gamma)$.

WCF comes from the definition.

$$A_\ell = \exp\Big(\sum_{Z(\gamma)\in \ell} a(\gamma) \Big)$$

$$A_V = \overrightarrow{\prod_{\ell \subset V}} A_\ell$$

Topology depends on stability data

Collection of elements

Think of it as a

Formalism for any wall crossing phenomena.

Walls of 2^{nd} kind $Z(\gamma) \in \mathbb{R}_+$.

KS is a generalization of Joyce to Triangulated case...

Another thing — relation to cplx integrable systems arising from 3d CY categories.

Formal auts of $\Gamma \otimes \mathbb{C}^*$ are gluing data

Meaning of cplx integrable

Fiber ~ abstract intermediate Jacobian

Basic ~ abstract cplx structures.

Seiberg–Witten etc.

Algebraic part related to motivic DT invts.

Noncommutative derived algebraic geom.

Ordinary alg. geom. is complicated.
After Bondal and Van den Bergh thm.
$\exists$ algebra

$$\mathcal{D}^b(X) \cong \mathcal{D}^b(A\text{-mod})$$

P – generator $\quad A = \mathrm{End}(P)$
Reduces to pure algebraic question
A_∞ - algebra + A_∞ - modules
So reduces to affine case.

By finding exceptional collection $\to$
quiver algebra $\to$ can say a lot.

From this point of view
Counting of BPS states should be

stated in a geometric independent way

3 diml CY $\mathcal{C}$ + stability condition

(algebra $\longrightarrow$ Spec(A) noncom.

$\hookrightarrow$ NC 3d CY with polarization.

Restriction on category

- compact $\Leftrightarrow$ algebraically Ext-finite

includes

e.g. $\mathcal{D}^b_Y(X)$ $X \sim$ local CY

support on Y.

- $(\cdot,\cdot)$: $\mathrm{Hom}(E,F) \otimes \mathrm{Hom}(F,E)$

$\longrightarrow k[-3]$

char$(k)=0$

"Serre duality"

- A_∞ category
- Δ- category (so we have shifts)

- W = superpotential in this abstract situation

$$W_N : \bigotimes_{i=1}^{N} \mathrm{Hom}(E_i, E_{i+1})[1] \longrightarrow k$$

$E_{N+1} = E_1$

defined using

$$W_N(a_1, \dots a_N) = (m_{N-1}(a_1, \dots a_{N-1}), a_N)$$

cyclically invt.

(Here is where 3 dim'l is important

- $E \in Ob(\mathcal{C}) \rightsquigarrow$

$$W_E^{tot}(\alpha) = \sum_{n \geq 2} \frac{W_n(\alpha, \dots, \alpha)}{n}$$

$$\alpha \in \mathrm{Hom}^{*}(E, E][1]$$

$W|_{\mathrm{Hom}^1} = W_E$ = potential of E

generalized Chern-Simons.

If Category is minimal

$\mathrm{Hom}^1(E,E) \simeq \underbrace{\mathrm{Ext}^1(E,E)}_{\text{tangent space of deformations}}$

$W_E(\alpha)$ formal function on $\mathcal{M}_E$.

- Orientation data on $\mathcal{C}$

kind of choice of $\sqrt{\det \mathrm{Ext}^\bullet(E,E)}$

$\det \mathrm{Ext}^\bullet(E,E)$ = superline bundle on the moduli space.

(Comes from existence of the product m_3 .....)

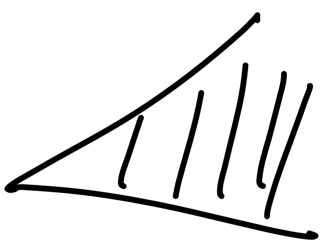
You will need it to do ---

- $H(\mathcal{C})$ — motivic Hall algebra

we assume objects form countable union of constructible sets and $\exists$ formalism of motivic

$H(\mathcal{C})$ associative algebra on objects.

Valerio's talk gave example.



V $\quad \mathcal{C}_V$ subcategory of extensions of s.s. objects E with $Z(E) \in V$

$$A_V^{Hall} \in H(\mathcal{C}_V)$$

$$A_V^{Hall} = \sum_{[E] \in Isom(\mathcal{C}_V)} [E]$$

Then you check

$$\boxed{A_V^{Hall} = A_{V_1}^{Hall} A_{V_2}^{Hall}}$$

Equivalent to existence of a collection

Note: In case of $\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$

equivalent to existence of $a(\gamma)$

Not very deep.

More importantly, using $W(\alpha)$

Can define a homomorphism

from Hall algebra to the quantum torus.

Given CY category w/ stability data

$$\mathrm{ch}: K_0(\mathcal{C}) \longrightarrow \Gamma$$ "Chern character"

$$\langle E, F \rangle = \chi(E, F)$$

$$= \sum (-1)^i \dim \mathrm{Ext}^i(E, F)$$

Skew symmetric.

not necessarily nondegenerate

$(\Gamma, \langle -, \cdot \rangle) \Rightarrow$ can define

the quantum torus $R_{\Gamma, \langle , \rangle}$

$$\hat{e}_{\gamma_1} \hat{e}_{\gamma_2} = q^{\frac{1}{2}\langle \gamma_1 \gamma_2 \rangle} \hat{e}_{\gamma_1 + \gamma_2}$$

<u>Thm</u>: $\exists$ homomorphism

$$H(\mathcal{C}) \longrightarrow \underbrace{R_{\Gamma,\langle\rangle}}$$

coefficient ring is ring of motivic functions

Coefficients are motivic functions

Could just take rational functions of q. — then a thm

$$A_V^{Hall} \longrightarrow \sum_{E\in\mathcal{C}_V} \frac{W(E)}{[\mathrm{Aut}(E)]} := A_V^{mot}$$

$$A_V^{mot} \subset R_{\Gamma,\langle\rangle}$$

$W(E)$ = weights defined in terms of (motivic) Milnor fiber of W_E

$[\mathrm{Aut}(E)]$ = motive of gp of auts

$\therefore$ since alg. homomorphism

$$A_V^{mot} = A_{V_1}^{mot} A_{V_2}^{mot}$$

Collection of (A_V^{mot}) are the motivic DT invariants.

(? $\exists$ integration map on uncompleted original Hall algebra.)

Want to take a limit

$$e_{\gamma_1} e_{\gamma_2} = q^{\frac{1}{2}\langle \gamma_1, \gamma_2 \rangle} e_{\gamma_1+\gamma_2}$$

$q^{1/2} \longrightarrow -1$ to be compatible with Behrend's counting function

! Limit of A_V^{mot} does not exist.

Conjecture

$$\lim_{q^{1/2}\to -1} A_V^{mot}(\cdot)(A_V^{mot})^{-1} = A_V(\cdot)A_V^{-1}$$

* Orientation data were used in the definition of the motivic weight like taking squareroot of $\det\bar{\partial}$.

$$\lim_{q^{1/2}\to -1} \mathrm{Ad}_{A_V^{mot}} = \text{Poisson diffeo of the torus}$$

$$= \prod_{Z(\gamma)\in V} T_\gamma^{\Omega(\gamma)}$$

$$T_\gamma(e_\mu) = (1-e_\gamma)^{\langle\gamma,\mu\rangle} e_\mu$$

$$= \exp\{-Li_2(e_\gamma), \cdot\} e_\mu$$

<u>Conjecture</u>: $\Omega(\gamma\ ) \in \mathbb{Z}$

these will be the numerical DT invariants.

How we see cluster transformations

$\mathcal{C}$ $(E_i)_{i\in I}$ s.t.

$$\mathrm{Ext}^\bullet(E_i, E_i) = H^\bullet(S^3)$$

$\mathrm{Ext}^m(E_i, E_j) \neq 0$ only for $m=1$ or $m=2$

(only one of 2)

$\Longrightarrow$ Quiver:

$$i \xrightarrow{a_{ij}} j$$

Moreover $\exists\, W_E \quad \forall E$

take it for $E = \bigoplus E_i$

$W \rightsquigarrow$ potential for Q

<u>Thm</u> $(Q, W) \longleftrightarrow \left(\begin{array}{l} 3d\ CY + \\ \text{such generators} \end{array} \right)$

Can define mutations at E_i.

tilting ; change of t structure.

$$(E_i) \rightsquigarrow (E_i')$$

can compute DT's $\quad A_V^{(E_i)} \qquad A_V^{(E_i')}$

ad symplectic

depend on derived category so don't depend on mutation

become auts of T

Change of coord's is a cluster trnsn.

Quiver: Choose vertex, say, 0

$$\bullet \longrightarrow 0 \xrightarrow{a_{0j}} \bullet_j$$

Permutation on this vertex

$$(E_i) \longrightarrow (E_i')$$

$$E_i' = E_i \quad i < 0$$

$$E_0' = E_0[-1]$$

$$E_i' = \mathrm{Cone}\left(E_0 \otimes \mathrm{Ext}^\bullet(E_0, E_i) \to E_i\right)$$

$$i > 0$$

K_0 theoretically it is just reflects

$$v_i = [E_i]$$

Euler matrix is a mutation.

$$a_{ij} \longrightarrow a_{ij}'$$ standard mutation

$\exists$ orientation data for CY's w/ cluster. . . but depends on choice of gen's.

Need to double the lattice

$$\Gamma \oplus \Gamma^\vee$$

(it could happen that $K_0(\mathcal{C}) = \mathbb{Z}$)

$\exists$ corresponding quantum torus

$R_{\Gamma \oplus \Gamma^\vee}, \langle , \rangle$

e_i $i \in I$ but also $e_i^\vee$

$\mathcal{E} = (E_i)_{i \in I}$

Consider the element $A_\mathcal{E}$ corresponding to A_V^{mot} in the limit $q^{1/2} \to -1$.

Assume $\mathrm{Ad}_{A_\mathcal{E}}$ is a birational symplectomorphism $T_{\Gamma \oplus \Gamma^\vee} \circlearrowleft$

$$T_{\Gamma \oplus \Gamma^\vee} = (\Gamma \oplus \Gamma^\vee) \otimes \mathbb{C}^*$$

Make a mutation on one vertex

$$\mathcal{E} = (E_i)_{i\in I} \xrightarrow{M_{E_0}} \mathcal{E}' = (E_i')_{i\in I}$$

Coordinates x_i, y_i coords on $T_{\Gamma\oplus\Gamma^\vee}$

$y_i \leftrightarrow e_i = (0, \dots, 1, \dots 0)$

$x_i \leftrightarrow e_i^\vee$

$\tau: \quad x_i \to x_i^{-1}, \; y_i \to y_i^{-1}$

$$\mathrm{Ad}_{A_\mathcal{E}} \circ \tau = \phi_\mathcal{E} : \overset{\text{"}T\text{"}}{T_{\Gamma\oplus\Gamma^\vee}} \circlearrowleft$$

<u>Proposition</u>: $\exists$ explicit map

$$C_0 \circ \phi_\mathcal{E} = \phi_{\mathcal{E}'} \circ C_0$$

C_0: birational automorphism of $T \circlearrowleft$

where

$$C_0 : \; y_i \longmapsto \frac{y_i'}{\left(1 - \frac{1}{y_0'}\right)^{a_{i0}}} \qquad i < 0$$

$$y_0 \longrightarrow (y_0')^{-1}$$

$$y_i \longmapsto y_i'(1 - y_0')^{a_{0i}} \qquad i > 0$$

with similar formulae for x_i

So we can construct a cluster variety in the sense of Fock + Goncharov. Including quantum cluster variety.