Soibelman – Wall-crossing

Reminder:

If all you care about is WCF you can just think about Lie algebras:

$$\mathfrak{g} = \bigoplus_{\gamma} \mathfrak{g}_\gamma \qquad \text{Stability data: } Z : \Gamma \to \mathbb{C}$$

$$a = (a_\gamma) \in \mathfrak{g}_\gamma \quad \forall \gamma \neq 0$$

$$a_\gamma \neq 0 \Rightarrow \|\gamma\| \leq C\,|Z(\gamma)|$$

$\exists$ Hausdorff topology on $\mathrm{Stab}(\mathfrak{g})$

Ex $\mathfrak{g}_\gamma = \mathbb{Q}\cdot e_\gamma \quad [e_{\gamma_1}, e_{\gamma_2}] = (-)^{\langle\gamma_1,\gamma_2\rangle} e_{\gamma_1+\gamma_2} \qquad a(\gamma) \leftrightarrow \Omega(\gamma)$

[The WCF really "comes from" this Hausdorff topology on $\mathrm{Stab}(\mathfrak{g})$ — ask when a path in $\mathrm{Stab}(\mathfrak{g})$ will be continuous]

NC derived algebraic geometry

Bondal–van den Bergh: $D^b(X) \simeq D^b(A\text{-modules})$ (in the A_∞ sense)

$D^b(X)$: generated by P; $A = \mathrm{End}(P)$

So you "only" have to think about A_∞ algebras...

3CY category + stability condition $\longleftrightarrow$ NC 3dCY with polarization

compact $\longleftrightarrow$ Ext-finite

$$(\,,\,) : \mathrm{Hom}(E,F) \otimes \mathrm{Hom}(F,E) \to k[-3]$$

$$W_N : \bigotimes_{i=1}^{N} \mathrm{Hom}(E_i, E_{i+1})[1] \to k$$

$$W_N(a_1,\dots,a_N) = (m_{N-1}(a_1,\dots,a_{N-1}), a_N) \qquad \mathbb{Z}/N\text{ - invariant}$$

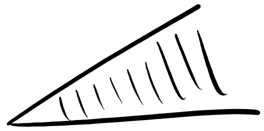
$$E \in \mathrm{Ob}(C) \rightsquigarrow W_E^{tot}(\alpha) = \sum_{n\geq 2} \frac{W(\alpha,\dots,\alpha)}{n} \qquad \alpha \in \mathrm{Hom}(E,E)[1]$$

"generalized Chern–Simons"

W_E is "formal function on $\mathcal{M}_E$"

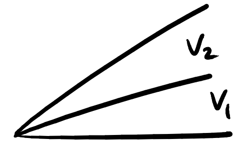
Also need <u>orientation data</u> ie $\sqrt{\det \mathrm{Ext}^1(E,E)}$

<u>Then</u>: $H(\mathcal{C})$ "motivic Hall algebra"



$\mathcal{C}_V = \{$extensions of ss objects E with $Z(E) \in V\}$

$A_V^{Hall} = A_{V_1}^{Hall} A_{V_2}^{Hall}$



Note: In case $\mathfrak{g} = \oplus \mathfrak{g}_\gamma$ giving $A_V \longleftrightarrow$ giving (a_γ)

$\mathrm{ch}: K_0(\mathcal{C}) \longrightarrow \Gamma \qquad \langle E,F\rangle = \chi(E,F) = \sum (-)^i \dim \mathrm{Ext}^i(E,F)$

Quantum tori: $\hat{e}_{\gamma_1} \cdot \hat{e}_{\gamma_2} = q^{\frac{1}{2}\langle \gamma_1, \gamma_2\rangle} \hat{e}_{\gamma_1+\gamma_2}$ <u>generate</u> $R_{\Gamma,\langle,\rangle}$

Using W can define a <u>homomorphism</u>

$$H(\mathcal{C}_V) \longrightarrow R_{\Gamma,\langle,\rangle}$$

$$A_V^{Hall} \longmapsto \sum_{E \in \mathcal{C}_V} \frac{w(E)}{[\mathrm{Aut}\, E]} = A_V^{mot}$$

[$w(E)$ defined in terms of <u>Milnor fiber</u> of W_E, <u>using</u> the orientation data...]

$\{A_V^{mot}\}$ is <u>motivic</u> Donaldson–Thomas invariants.

<u>Conjecture</u>: $\lim\limits_{q^{\frac{1}{2}} \to -1} Ad(A_V^{mot})$ <u>exists</u>.

It gives a (formal) Poisson diffeomorphism of the tori. Extract $\Omega(\gamma)$ from this.

<u>Conjecture</u>: $\Omega(\gamma)$ exists. (and <u>independent</u> of orientation?)

Relation to clusters:

$\mathcal{C}$ $\quad (E_i)_{i\in I}$

$\mathrm{Ext}^\bullet(E_i, E_i) = H^\bullet(S^3)$

$\mathrm{Ext}^m(E_i, E_j) \neq 0$ for $m = 1$ or 2

$\chi(E_i, E_j) = -a_{ij}$

From this stuff define Q and W.

$$i \longrightarrow 0 \xrightarrow{a_{0j}} j$$

Thm: $(Q, W) \longleftrightarrow$ (3d CY + such generators)

Mutations at E_{i0}: $\quad (E_i) \rightsquigarrow (E_i')$

$E_i' = E_i \quad (i < 0)$

$E_0' = E_0[-1]$

$E_i' = \mathrm{Cone}\left(E_0 \otimes \mathrm{Ext}^1(E_0, E_i) \to E_i\right)$

Then the a_{ij} transform by mutation $[Q \to Q']$

Double the lattice: $\Gamma \oplus \Gamma^\vee \rightsquigarrow R_{\Gamma\oplus\Gamma^\vee}, \langle , \rangle$

$l_i \quad l_i^\vee$

Consider the element $A_\mathcal{E}$ [corresp. to A_V^{mot} in limit $q^{1/2} \to -1$]

Assume it's birational symplectomorphism of doubled torus T (with doubled coords x_i, y_i)

Now mutate one vertex.

$\tau: x_i \mapsto x_i^{-1}$

$y_i \mapsto y_i^{-1}$

$A_V^{(E)} \qquad A_V^{(E')}$ both define $T \circlearrowleft$

$\mathrm{Ad}_{A^{(E)}} \circ \tau = \varphi_E$

Prop: $C_0 \circ \varphi_E = \varphi_{E'} \circ C_0$ where $C_0 : T \circlearrowleft$

$\underline{C_0}$:

$$y_i \mapsto \frac{y_i'}{\left(1-\frac{1}{y_0'}\right)^{a_{i0}}} \quad i<0$$

$$y_0 \longmapsto (y_0')^{-1}$$

$$y_i \longmapsto y_i(1-y_0')^{a_{i0}}$$

[similarly for x_i]