Szendrői – Refined topological vertex & Poincaré polynomials

DT/GW are Euler characteristics of moduli spaces, in some sense to be described here

Refine/deform $\Longrightarrow$ Poincaré polynomial
"BPS state algebra"

X CY_3 $\longrightarrow$ DT invariants and their generating series

$$Z^X(q,t) = \sum_{\substack{n\in\mathbb{Z}\\ \beta\in H_2(X)}} DT^X(n,\beta)\, q^n\, T^\beta$$

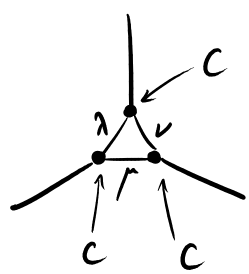
$$DT^X(n,\beta) = \deg\left[\mathcal{M}^X(n,\beta)\right]^{vir} \in \mathbb{Z}$$

[moduli space of ideal sheaves of subschemes $I_Z \subset \mathcal{O}_X$]

$$\left[\mathrm{supp}(\mathcal{O}/I)\right] = \beta \in H_2(X)$$

Thm $\quad Z_0^X(q) = M(-q)^{\chi(X)}$ $\quad$ (X compact)

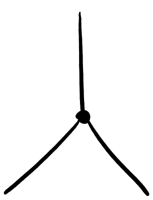
X toric CY_3 — not compact but $DT^X(n,\beta)$ can still be defined for compactly supported β, by torus localization. Computable from toric diagram of X using the topological vertex as basic ingredient.



C can be computed using McMahon and relative Schur functions

$C_{\lambda\nu\phi}(q)$

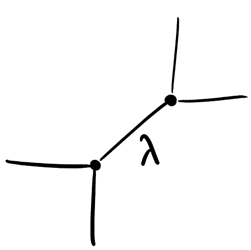
ex 1



$$Z^{\mathbb{C}^3}(q) = C_{\phi\phi\phi}(q) = M(-q) \qquad \text{from torus loc. on } \mathrm{Hilb}(\mathbb{C}^3)$$

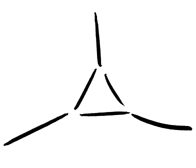
[to prove $\prod (1-(-q)^n)^{-n} = \sum_\alpha (-q)^{|\alpha|}$
use operator formalism, or something...]

ex 2



$$Z^{\mathcal{O}_{\mathbb{P}^1}(-1,-1)} = M(-q)^2 \prod_n \left(1-(-q)^n T\right)^n$$

ex 3

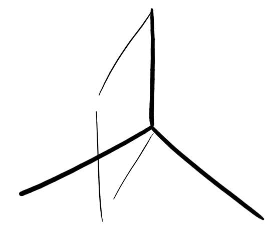


$Z^{\mathcal{O}_{\mathbb{P}^2}(-2)}$ has no pretty infinite product...

__IKV__: __deformation__ of the topological vertex

$$Z^{\mathbb{C}^3}(x,y) = \sum_{\alpha} x^{|\alpha|_+} y^{|\alpha|_- + |\alpha|_-}$$
$$= \prod_{i,j\geq 1} (1 - x^{i-1} y^{j})^{-1}$$

α a 3-d Young diagram which we __chop__ by putting a plane in the middle



$\alpha_{\pm}$ count # boxes on the two sides of this plane

Modify this by

$$Z^{\mathbb{C}^3}(x,y)^{(k)} = \sum_{\alpha} x^{|\alpha|_+ + k|\alpha|_0} y^{|\alpha|_- + (1-k)|\alpha|_0} \qquad k \in \tfrac{1}{2}\mathbb{Z}$$
$$= \prod_{i,j\geq 1}^{\alpha} (1 - x^{i-1+k} y^{j-k})^{-1}$$

For $X = $ local $\mathbb{P}^1$,
$$\frac{Z^{\mathbb{P}^1}(x,y,T)}{Z_0^{\mathbb{P}^1}(x,y,T)} = \prod_{i,j\geq 1} (1 - x^{i-\frac{1}{2}} y^{j-\frac{1}{2}} T)$$

$$Z^{\mathbb{C}^3}(q,t) = \sum_{n} q^n P_n(t)$$

$\mathrm{Hilb}(\mathbb{C}^3, n) \longrightarrow$

- $n=1$ $\mathbb{C}^3$
- $n=2,3$ smooth
- $n=4,\dots,8$ or 9 irred
- large n $\dim \sim O(n^{4/3})$ $\leftarrow$ compare to $S^n(\mathbb{C}^3)$ which has $\dim = 3n$!

__Fact__: $\mathrm{Hilb}(\mathbb{C}^3, n) \subset \mathbb{C}^n \times M_n(\mathbb{C})^{\oplus 3} /\!\!/ GL(n)$ is __critical__ locus of $W = \mathrm{Tr}(ABC - ACB)$ (in scheme-theoretic sense)

with some stability condition

($dW=0$ implies A,B,C all __commute__; so we get $A,B,C \in M_n(\mathbb{C})$ and $\underline{v} \in \mathbb{C}^n$ such that A,B,C commute and $\langle A,B,C\rangle \underline{v} = \mathbb{C}^n$)

__Because__ of this realization of $\mathrm{Hilb}(\mathbb{C}^3,n)$, __there is__ $\Phi_W \in \mathrm{Perv}_c(\mathrm{Hilb}(\mathbb{C}^3,n))$

"perverse sheaf of vanishing cycles"

Prop: X a complex 3-fold $\Rightarrow$ gluing such sheaves from local analytic patches gives a canonical perverse sheaf $\Phi_n \in \mathrm{Perv}_c(\mathrm{Hilb}(X,n))$

[Rk: $\deg [\mathrm{Hilb}(\mathbb{C}^3,n)]^{vir}$
$\quad \| $
$\chi(\mathbb{H}^\bullet(\mathrm{Hilb}(\mathbb{C}^3,n), \Phi_W))$]

Main conjecture:

$$\sum_n q^n \sum_k t^k \dim \mathbb{H}^k(\mathrm{Hilb}(X,n), \Phi_n)$$

$$\Big\| \; \begin{smallmatrix} x = qt \\ y = q/t \end{smallmatrix}$$

$$\prod_{i,j \geq 1} \frac{(1 - x^{i+1/2} y^{j-3/2})^{b_1(X)} (\text{—})^{b_3(X)} (\text{—})^{b_5(X)}}{(\text{—})^{b_2(X)} (\text{——})^{b_4(X)} (\text{—})^{b_6(X)}}$$

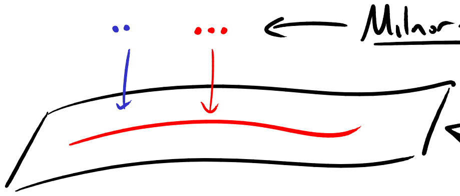
(cf. 2-d case, Göttsche)

[So this is a conjecture for the deformed degree zero part of DT theory for X.]

If $x = y$ reduce to $M(q)^{\chi(X)}$ so it works [modulo some tricky signs!]

So $\bigoplus \mathbb{H}^k(\mathrm{Hilb}(X,n), \Phi_n)$ is a module for some BPS state algebra?

The underlying picture of the ordinary DT invariants here is



$\longleftarrow$ Milnor fiber (vanishing cycle of W as approach $dW=0$)

$\longleftarrow$ D-brane moduli space $[dW=0]$

DT inv't is a $\sum$ over strata. Each stratum weighted by

$$\chi(\text{stratum}) \cdot \left[1 - \chi(\text{Milnor fiber})\right] \cdot (-)^{\dim(\text{Milnor fiber})}$$

[Don't have a similarly simple picture for the refined DT invariant]