Toledano Laredo - Stability conditions and Stokes factors

Motivation: Joyce on hol gen func for invariants of abelian categories

Dictionary: Stability conditions $\longleftrightarrow$ Stokes for irreg connections on $\mathbb{CP}^1$

Stability conditions

$\mathcal{D}$ Δ'd category

Def A central charge Z on $\mathcal{D}$ is a gp hom $K(\mathcal{D}) \to \mathbb{C}$

Ex $\mathcal{D} = D^b(\mathrm{Coh}\, X)$, X smooth proj curve over $\mathbb{C}$

$$Z(E) = -\deg E + i \operatorname{rk} E$$

For $E \in \mathrm{Coh}\, X$, $\phi(E) = \frac{1}{\pi} \operatorname{phase} Z(E) \in (0,1]$

Ex $\mathcal{D} = D^b \mathrm{Fuk}(M, I, B + i\omega)$, M CY 3-fold

$$Z_I(L) = \int_L \Omega_I$$

L slag of phase $\phi \Rightarrow Z_I(L) \in \mathbb{R}_+ e^{i\pi\phi}$

Def A stability condition on $\mathcal{D}$ consists of:

1) A central charge Z on $\mathcal{D}$
2) $\forall \phi \in \mathbb{R}$, a full additive subcategory $\mathcal{P}(\phi)$ "semistable objects of phase ϕ"

s.t.

a. $E \in \mathcal{P}(\phi) \setminus 0 \Rightarrow Z(E) \in \mathbb{R}_+ e^{i\pi\phi}$

b. $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$

c. $\phi_1 > \phi_2$, $E_i \in \mathcal{P}(\phi_i) \Rightarrow \mathrm{Hom}_{\mathcal{D}}(E_1, E_2) = \phi$

d. $\forall E \in \mathcal{D}$, $\exists \phi_1 > \cdots > \phi_n$ and a collection of Δ

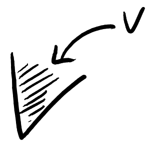
$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_n = E \arrow[dl] \\
A_1 \arrow[u, dashed] & & A_n \arrow[u, dashed] &
\end{tikzcd}$$

$A_i \in \mathcal{P}(\phi_i)$

$$\begin{array}{c} \mathrm{Stab}(\mathcal{D}) = \{ (Z, \mathcal{P}) \mid \mathcal{P} \text{ locally finite} \} \\ \downarrow Z \\ \mathrm{Hom}(K(\mathcal{D}), \mathbb{C}) \end{array}$$

Thm For each connected component Σ of $\mathrm{Stab}(D)$, there is a linear subspace V_Σ of $\mathrm{Hom}_{\mathbb{Z}}(K(D), \mathbb{C})$ with a well-defined topology s.t. $Z: \Sigma \to V_\Sigma$ is a <u>local</u> <u>homeomorphism</u>.

[Key ingredient:



$P_Z(V)$ = category gen by ss objects with phase lying in $V = P_{Z'}(V)$ provided nothing goes in or out of the sides]

<u>Alternative characterization</u>

<u>Def</u>: A <u>stability function</u> on an abelian category is a gp hom

$$Z: K(A) \longrightarrow \mathbb{C}$$
$$\cup$$
$$K_{>0}(A) \longrightarrow \mathbb{H} = \{re^{i\pi\phi} \mid r>0,\ 0<\phi\leq 1\}$$

<u>Def</u> 1) $E \in A \setminus 0$, $\phi(E) = \frac{1}{\pi}\arg Z(E) \in (0,1]$

2) E is semistable if $\forall\, 0 \neq F \subset E$, $\phi(F) \leq \phi(E)$

3) Z has the <u>HN property</u> if $\forall E \subset A$

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

s.t. $F_i = E_i / E_{i-1}$ and $\phi(F_1) > \phi(F_2) > \cdots$

<u>Def</u> A <u>stability condition</u> on A is given by a stability function satisfying the HN property.

<u>Ex</u> $A = \mathrm{Rep}(R)$, R f-d algebra over $\mathbb{C}$

$K(A) \simeq \mathbb{Z}[S_1] \oplus \cdots \oplus \mathbb{Z}[S_N]$ S_i simple modules $\mathrm{Stab}(A) \simeq \mathbb{H}^N$

<u>Thm</u>: Giving a stab. cond. on D

$$\Updownarrow$$

Giving 1) a bounded t-structure on D, with heart A

2) a stability condition Z on A

<u>Why?</u> $(Z, P) \longrightarrow A$ = cat gen. by $P((0,1])$, $Z = Z|_A$

$(A, Z) \longrightarrow P(\phi) = \begin{cases} \{\text{semistable objects in } A \text{ of phase } \phi\} & \text{if } \phi \in (0,1) \\ P(\phi - [\phi])[\phi] & \text{in general} \end{cases}$

Hall algebras

$\mathcal{A}$ abelian category
$\mathcal{M}$ stack of objects of $\mathcal{A}$

e.g. $\mathcal{A} = Rep(R)$ $\quad R$ finite-dimensional over $\mathbb{C}$

$$\mathcal{M} = \coprod_{d \geq 0} Rep_d(R) / GL_d(\mathbb{C})$$

$H(\mathcal{A})$ = Hall algebra of $\mathcal{A}$ = {constructible functions on $\mathcal{M}$}

$$\begin{array}{ccccccc} \mathcal{M} & \times & \mathcal{M} & \times & \mathcal{M} & \subset & \mathcal{M}^{(3)} = \{0 \to A \to B \to C \to 0\} \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_3} & & \\ \mathcal{M} & & \mathcal{M} & & \mathcal{M} & & \end{array}$$

Product structure by <u>convolution</u> $\quad f * g(C) = \int_{0 \to A \to B \to C \to 0} f(A) \cdot g(B)$ $\quad$ (associative)

(Euler characteristic)

1_0 = characteristic function of the zero object

<u>Reineke</u>:

$\mathbb{1}_{\mathcal{A}} \in H(\mathcal{A})$ characteristic function of <u>all</u> objects

SS_ℓ = char. f^n. of all <u>semistable</u> objects with $Z(E) \in \ell$ $\quad$ ($\ell \subset \mathbb{C}^\times$ a ray)

HN property $\Leftrightarrow$ $\mathbb{1}_{\mathcal{A}} = \overrightarrow{\prod_{\ell \in \mathbb{H}}} SS_\ell$

already a kind of WCF — the RHS looks as if it depends on a stability condition, while the LHS obviously doesn't

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A little more about $H(\mathcal{A})$:

$\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ $\qquad \Delta$ co-comm $\implies$ $\mathcal{H} = U(n)$ a pronilpotent Lie algebra

$\Delta f(M, N) = f(M \oplus N)$

$$\Delta(\mathbb{1}_{\mathcal{A}}) = \mathbb{1}_{\mathcal{A}} \otimes \mathbb{1}_{\mathcal{A}}$$

$$\Delta(SS_\ell) = SS_\ell \otimes SS_\ell$$

Ex: R = quiver of finite Dynkin type $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ $\mathfrak{n} = \mathfrak{n}_+$

Stokes: [as in his previous talk]

Dictionary:

Stab	Hall	ODE
$(\mathcal{A}, Z)$	$\mathbb{1}_{\mathcal{A}}, Z$	S_+ (S_-)
$(\mathcal{P}, Z)$	SS_ℓ	Stokes factor
HN	$\mathbb{1}_{\mathcal{A}} = \overrightarrow{\prod} SS_\ell$	$S_+ = \overrightarrow{\prod} S_\ell$
wall cross	$\overrightarrow{\prod} SS_\ell^Z = \overrightarrow{\prod} SS_\ell^{Z'}$	Isomonodromy

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Application to Joyce's work:

$\mathcal{A}$ "good" abelian cat, $\mathcal{A} = \mathrm{Rep}(R)$

Z stab cond on $\mathcal{A} = \mathrm{Hom}(K(\mathcal{A}), \mathbb{C}) = \mathfrak{h}$

$\mathcal{H}$ = Hall alg of $\mathcal{A} = \mathcal{U}\mathfrak{n} \otimes \mathfrak{h}$

$\mathcal{B}$ prosolvable algebraic group with Lie algebra $\mathfrak{n} \rtimes \mathfrak{h}$

$$\nabla_{\mathcal{A},Z} = d - \left(\frac{Z}{t^2} + \frac{f}{t}\right) dt$$

Thm: TFAE:

1) $$f_\alpha = \sum_{n \geq 1} \sum_{\alpha_1 + \cdots + \alpha_n = \alpha} J_n(Z(\alpha_1), \ldots, Z(\alpha_n)) \, \mathcal{E}_{\alpha_1} * \cdots * \mathcal{E}_{\alpha_n}$$

$\mathcal{E}_{\alpha_i}$ is closely related to char f^n of semistable objects in class $\alpha_i \in K_{>0}(\mathcal{A})$

$J_n : (\mathbb{C}^\times)^n \to \mathbb{C}^\times$ are universal f^n's built by Joyce

2) Stokes matrices of $\nabla_{A,Z}$ are $S_+ = \mathbb{1}_A$
$S_- = \mathbb{1}_0$

3) Stokes factor S_ℓ of ∇ is $S_\ell = SS_\ell$

$$df_\alpha = \sum_{\beta+\gamma=\alpha} [f_\beta, f_\gamma]\, d\log\frac{\beta}{\gamma}$$
