Factorization of Ising Form Factors

Michael Assis

Stony Brook University

SCGP Workshop Jan 2010

Outline

- Diagonal Ising Form Factors
- Factorization in Maple
- Analytic Proofs
  - $f_0^{(2)} = \frac{1}{2} K(K - E)$
  - $f_N^{(2)}$ recurrence relation, N>0
  - $f_1^{(2)} = \frac{1}{2} - \frac{3}{2} KE - \frac{(t-2)}{2} K^2$
- Further Work

Diagonal Form Factors

Ising diagonal correlation functions can be written as a form factor expansion:

$$C(N,N) = \begin{cases} (1-t)^{1/4} \sum_{n=0}^{\infty} f_N^{(2n)} \quad , & T < T_c \\ (1-t)^{1/4} \sum_{n=0}^{\infty} f_N^{(2n+1)} , & T > T_c \end{cases}$$

$$t = \begin{cases} (\sinh 2K^v \sinh 2K^h)^{-2} \quad , & T < T_c \\ (\sinh 2K^v \sinh 2K^h)^{2} \quad , & T > T_c \end{cases}$$

Each $f_N^{(n)}$ is an n-fold integral

Examples

$$f_N^0 = 1$$

$$f_N^{(1)} = \frac{t^{N/2}}{\pi} \int_0^1 dx \frac{x^{N-1/2}}{\sqrt{(1-tx)(1-x)}}$$

$$f_N^1 = t^{N/2} \frac{\Gamma(N+1/2)}{\pi^{1/2} N!} {}_2F_1\left(\tfrac{1}{2}, N+\tfrac{1}{2}; N+1; t\right)$$

$$f_N^{(2)} = \frac{t^{N+1}}{\pi^2} \int_0^1 dx \int_0^1 dy \frac{x^{N+1/2} y^{N-1/2} \sqrt{(1-y)(1-ty)}}{(1-txy)^2 \sqrt{(1-x)(1-tx)}}$$

Computer Assisted Solutions

- Produce series in t
- Use Maple’s SeriestoDiffeq function to find ODE of which the series is a solution
  - Need sufficient terms, hundreds to thousands
- Factorize differential operator
- Construct solutions from operators

Example for $f_N^{(2)}$

$$O_N^{(2)} f_N^{(2)} = 0$$

$$O_N^{(2)} = L_3(N) \cdot L_1(N) \cong M_3(N) \oplus L_1(N) \cong Sym^2(L_2(N)) \oplus L_1(N)$$

$$L_1(N) = Dt$$

$$L_3(N) = Dt^3 + 4\frac{(2t-1)}{t(t-1)} Dt^2 + \frac{(14t^2-15t+2)}{t^2(t-1)^2} Dt + \frac{8t^2-15t+5}{2t^2(t-1)^3} - N^2\left(\frac{Dt}{t^2} + \frac{1}{t^3}\right)$$

$$f_0^{(2)} = \frac{1}{2} K(K-E)$$

$$f_1^{(2)} = \frac{1}{2} - \frac{3}{2} KE - \frac{(t-2)}{2} K^2$$

$$f_2^{(2)} = 1 + \frac{(t+1)}{3t} E^2 + \frac{(15t-4)}{6t} KE - \frac{(6t^2-11t+2)}{6t} K^2$$

Analytic Proofs

$$f_N^{(2)} = \int_0^1 dx \int_0^1 dy \frac{t^{N+1}}{\pi^2} \frac{x^{N+1/2} y^{N-1/2}}{(1-txy)^2} \frac{(1-y)^{1/2}(1-ty)^{1/2}}{(1-x)^{1/2}(1-tx)^{1/2}}$$

Integration by parts in y:

$$u = y^{N-1/2}(1-y)^{1/2}(1-ty)^{1/2} \qquad dv = \frac{dy}{(1-txy)^2}$$

$$du = Ny^{N-3/2}(1-y)^{1/2}(1-ty)^{1/2} - \frac{y^{N-3/2}(1-ty^2)}{2(1-y)^{1/2}(1-ty)^{1/2}} dy \qquad v = \frac{y}{(1-txy)}$$

N=0:

$$= \int_0^1 dx \int_0^1 dy \frac{t}{2\pi^2} \frac{x^{1/2}}{y^{1/2}} \frac{(1-ty^2)}{(1-x)^{1/2}(1-tx)^{1/2}(1-y)^{1/2}(1-ty)^{1/2}(1-txy)}$$

$$\frac{x^{1/2}}{y^{1/2}} ty^2 \quad x \leftrightarrow y \quad \frac{y^{1/2}}{x^{1/2}} tx^2 = \frac{x^{1/2}}{y^{1/2}} txy$$

$$= \int_0^1 dx \int_0^1 dy \frac{t}{2\pi^2} \frac{x^{1/2}}{y^{1/2}} \frac{1}{(1-x)^{1/2}(1-tx)^{1/2}(1-y)^{1/2}(1-ty)^{1/2}} = \frac{K}{2}(K-E) = \frac{t^{1/2}}{2} f_0^{(1)} f_1^{(1)}$$

Recurrence Relation

Using same method of swapping variables above,

$$f_N^{(2)} = \int_0^1 dx \int_0^1 dy \, \frac{t^{N+1}}{2\pi^2} \frac{x^{N+1/2} y^{N-1/2}}{(1-x)^{1/2}(1-tx)^{1/2}(1-y)^{1/2}(1-ty)^{1/2}} -$$

$$- \int_0^1 dx \int_0^1 dy \;\; N \frac{t^{N+1}}{\pi^2} \frac{x^{N+1/2} y^{N-1/2} (1-y)^{1/2} (1-ty)^{1/2}}{(1-x)^{1/2}(1-tx)^{1/2}(1-txy)}$$

From definition of $f_N^{(2)}$, the following can be recognized

$$f_N^{(2)} - f_{N+1}^{(2)} = \int_0^1 dx \int_0^1 dy \, \frac{t^{N+1}}{\pi^2} \frac{x^{N+1/2} y^{N-1/2} (1-y)^{1/2} (1-ty)^{1/2}}{(1-x)^{1/2}(1-tx)^{1/2}(1-txy)}$$

which is the second integral, giving

$$f_N^{(2)} = \int_0^{\text{`}} dx \int_0^1 dy \, \frac{t^{N+1}}{2\pi^2} \frac{x^{N+1/2} y^{N-1/2}}{(1-x)^{1/2}(1-tx)^{1/2}(1-y)^{1/2}(1-ty)^{1/2}} - N\left(f_N^{(2)} - f_{N+1}^{(2)}\right)$$

Rearranging,

$$f_{N+1}^{(2)} = \frac{N+1}{N} f_N^{(2)} - \frac{t^{N+1}}{2\pi N} \frac{\Gamma(N+1/2)\Gamma(N+3/2)}{\Gamma(N+1)\Gamma(N+2)} F\left(-1/2, N+1/2; N+1; t\right) F\left(1/2, N+3/2; N+2; t\right)$$

Recurrence Solutions

Recurrence only valid for N>0

$$f_{N+1}^{(2)} = \frac{N+1}{N} f_N^{(2)} - \frac{t^{N+1}}{2\pi N} \frac{\Gamma(N+1/2)\Gamma(N+3/2)}{\Gamma(N+1)\Gamma(N+2)} F(1/2, N+1/2; N+1; t) F(1/2, N+3/2; N+2; t)$$

$$f_{N+1}^{(2)} = \frac{N+1}{N} f_N^{(2)} - \frac{t^{1/2}}{2N} f_N^{(1)} f_{N+1}^{(1)}$$

Knowing $f_1^{(2)}$, any $f_N^{(2)}$ can be constructed:

$$f_N^{(2)} = N f_1^{(2)} - N \frac{t^{1/2}}{2} \sum_{j=1}^{N-1} \frac{f_j^{(1)} f_{j+1}^{(1)}}{j(j+1)}$$

Finding $f_1^{(2)}$

After integration by parts,

$$f_N^{(2)} = \int_0^1 dx \int_0^1 dy \frac{t^{N+1}}{2\pi^2} \frac{x^{N+1/2} y^{N-1/2}}{(1-x)^{1/2}(1-tx)^{1/2}(1-y)^{1/2}(1-ty)^{1/2}} -$$

$$-\int_0^1 dx \int_0^1 dy \; N \frac{t^{N+1}}{\pi^2} \frac{x^{N+1/2} y^{N-1/2} (1-y)^{1/2}(1-ty)^{1/2}}{(1-x)^{1/2}(1-tx)^{1/2}(1-txy)}$$

Scale and re-write in contour form:

$$f_N^{(2)} = \frac{1}{(2\pi i)^2} \oint_C dx \oint_C dy \frac{x^{N+1/2} y^{N-1/2}}{2(x-\alpha_2)^{1/2}(1-\alpha_2 x)^{1/2}(y-\alpha_2)^{1/2}(1-\alpha_2 y)^{1/2}} -$$

$$x \to \frac{x}{\alpha_2} \qquad \alpha_2 = t^{1/2}$$

$$|x| = 1-\varepsilon$$

$$y \to \frac{y}{\alpha_2} \qquad |y| = 1-\varepsilon$$

$$-\frac{1}{(2\pi i)^2} \oint_C dx \oint_C dy N \frac{x^{N+1/2} y^{N-1/2}}{(1-xy)} \frac{(y-\alpha_2)^{1/2}(1-\alpha_2 y)^{1/2}}{(x-\alpha_2)^{1/2}(1-\alpha_2 x)^{1/2}}$$

Finding $f_1^{(2)}$ cont ...

Differentiate with respect to α_2 :

$$\frac{\partial f_N^{(2)}}{\partial \alpha_2} = \frac{1}{(2\pi i)^2} \oint_C dx \oint_C dy \frac{(1-\alpha_2{}^2)}{4} \frac{x^{N+1/2} y^{N-1/2} \left[2N(x-y)-(x+y)\right]}{(x-\alpha_2)^{3/2}(1-\alpha_2 x)^{3/2}(y-\alpha_2)^{1/2}(1-\alpha_2 y)^{1/2}}$$

Write as hypergeometric functions:

$$\frac{df_N^{(2)}}{dt} = \frac{(1-t)t^N}{4\pi\ \Gamma(N+1)\Gamma(N+2)} \Big[(2N+1)\Gamma(N+3/2)^2\ {}_2F_1\left(3/2, N+3/2; N+1; t\right) {}_2F_1\left(1/2, N+3/2; N+2; t\right) +$$

$$+ (-2N+1)\Gamma(N+5/2)\Gamma(N+1/2)\ {}_2F_1\left(3/2, N+5/2; N+2; t\right) {}_2F_1\left(1/2, N+1/2; N+1; t\right) \Big]$$

For N=1, write in E, K basis:

$$\frac{df_1^{(2)}}{dt} = \frac{1}{4t}\left(\frac{3}{t-1}K^2 + 2\frac{t-2}{t-1}KE - E^2\right) = \frac{d}{dt}\left(\frac{1}{2} - \frac{3}{2}KE - \frac{t-2}{2}K^2\right)$$

New Hypergeometric Identities

Expanding $\frac{1}{(1-txy)^2}$,

$$f_N^{(2)} = \int_0^1 dx \int_0^1 dy \frac{t^{N+1}}{\pi^2} \frac{x^{N+1/2} y^{N-1/2}}{(1-txy)^2} \frac{(1-y)^{1/2}(1-ty)^{1/2}}{(1-x)^{1/2}(1-tx)^{1/2}}$$

$$= \sum_{j=0}^{\infty} \frac{(j+1)(1/2)_{N+j}(3/2)_{N+j} t^{N+1+j}}{4(N+1+j)} {}_2F_1\left(-1/2, N+1/2+j; N+2+j; t\right) {}_2F_1\left(1/2, N+3/2+j; N+2+j; t\right)$$

$$\frac{1}{2} - \frac{3}{2} KE - \frac{t-2}{2} K^2 = \sum_{j=0}^{\infty} \frac{(j+1)(1/2)_{j+1}(3/2)_{j+1} t^{j+1}}{4(j+2)} {}_2F_1\left(-1/2, j+3/2; j+3; t\right) {}_2F_1\left(1/2, j+5/2; j+3; t\right)$$

Further Work

Understand $f_N^{(2)}$ solutions

- Simplify expressions
- Integrate $\frac{df_N^{(2)}}{dt}$ directly
- Understand the constant terms

Look at higher form factors, $f_N^{(n)}$, n=3, 4, …