



# Temperature correlation functions of quantum spin chains at magnetic and disorder field

Hermann Boos

Bergische Universität Wuppertal

joint works with: F. Göhmann, arXiv:0903.5043

M. Jimbo, T.Miwa, F. Smirnov, Y. Takeyama, arXiv:0801.1176

M. Jimbo, T.Miwa, F. Smirnov, arXiv:0903.0115



Main question: does factorization of correlation functions of the XXZ model first discovered for zero temperature work in the temperature case also?

Some important issues:

- Multiple integral representation works by Kyoto school (92–93) and Lyon group (98-2000)
- Suzuki-Trotter formalism and TBA approach to thermodynamics  
Suzuki (85), Klümper-Pearce-Batchelor (91-92), Destri-De Vega (93)...
- Generalization of multiple integrals to the temperature case  
Göhmman-Klümper-Seel (04)
- Observation of factorization of multiple integrals and relation to the qKZ  
Takahashi (77), Korepin-HB (01-02), Korepin-Smirnov-HB (03-04)
- Reduced qKZ, exponential form and fermionic basis  
Jimbo-Miwa-Smirnov-Takeyama-HB (05-09), Jimbo-Miwa-Smirnov (08)



- The Hamiltonian of the infinite spin- $\frac{1}{2}$  XXZ chain

$$\mathcal{H}_{\text{XXZ}} = \frac{1}{2} \sum_{j=-\infty}^{\infty} \left( \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z \right), \quad \Delta = \frac{1}{2}(q + q^{-1})$$

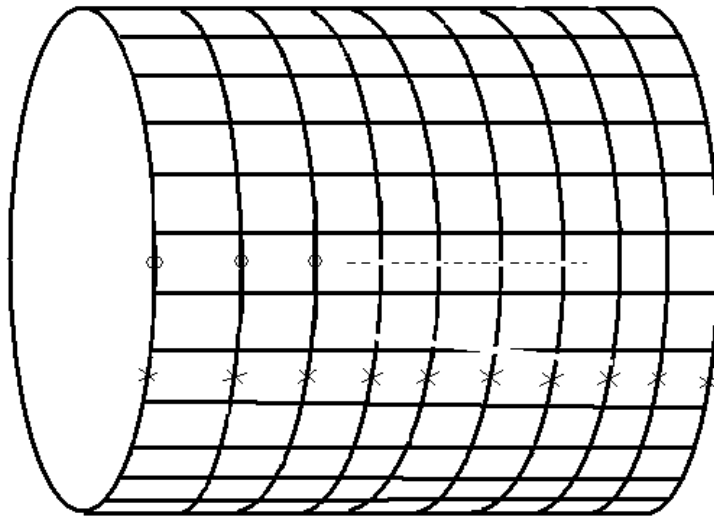
$$q = e^\eta = e^{\pi i \nu}$$

- Integrable structure is generated by  $R$ -matrix of the 6-vertex model

$$R(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{C}^2 \otimes \mathbb{C}^2$$

$$b(\lambda) = \frac{\text{sh}(\lambda)}{\text{sh}(\lambda + \eta)}, \quad c(\lambda) = \frac{\text{sh}(\eta)}{\text{sh}(\lambda + \eta)},$$

# Partition function of the 6-vertex model on cylinder



Cylinder is infinite  
in spacial direction

Cut corresponds to  
insertion of local operator  $\mathcal{O}$

We call  $q^{\alpha \sum_{j=-\infty}^0 \sigma_j^z}$   $\mathcal{O}$  quasi-local  
operator with tail  $\alpha$

$\kappa$  – magnetic field  
 $\alpha$  – disorder parameter

$$\begin{array}{c}
 q^{\alpha \sigma^z} \quad | \quad \mathcal{O} \\
 q^{\kappa \sigma^z} \quad | \quad *
 \end{array}
 \quad \text{Matsubara expectation values:}$$

$$Z^\kappa \left\{ q^{\alpha \sum_{j=-\infty}^0 \sigma_j^z} \mathcal{O} \right\} = \frac{\text{Tr}_S \text{Tr}_M \left( T_{S,M} q^{\kappa \sum_{j=-\infty}^{\infty} \sigma_j^z + \alpha \sum_{j=-\infty}^0 \sigma_j^z} \mathcal{O} \right)}{\text{Tr}_S \text{Tr}_M \left( T_{S,M} q^{\kappa \sum_{j=-\infty}^{\infty} \sigma_j^z + \alpha \sum_{j=-\infty}^0 \sigma_j^z} \right)}$$

$T_{S,M}$  is the transfer matrix with  $\mathfrak{H}_S = \bigotimes_{j=-\infty}^{\infty} \mathbb{C}^2$  and  $\mathfrak{H}_M = \bigotimes_{j=1}^n \mathbb{C}^2$

# Density Matrix Elements in standard Basis



Let us define the elements of density matrix taking  $\mathcal{O} = E_{\varepsilon'_1}^{\varepsilon_1} \cdots E_{\varepsilon'_n}^{\varepsilon_n}$

$$D_{N\varepsilon_1, \dots, \varepsilon_n}^{\varepsilon'_1, \dots, \varepsilon'_n}(\xi_1, \dots, \xi_n) := Z^{\mathbf{k}} \left\{ q^{\alpha \sum_{j=-\infty}^0 \sigma_j^z} E_{\varepsilon'_1}^{\varepsilon_1} \cdots E_{\varepsilon'_n}^{\varepsilon_n} \right\}$$

$$= \frac{\langle \mathbf{k} + \alpha | T_{\varepsilon_1}^{\varepsilon'_1}(\xi_1, \mathbf{k}) \cdots T_{\varepsilon_n}^{\varepsilon'_n}(\xi_n, \mathbf{k}) | \mathbf{k} \rangle}{T(\xi_1, \mathbf{k}) \cdots T(\xi_n, \mathbf{k}) \langle \mathbf{k} + \alpha | \mathbf{k} \rangle}$$

with unique eigenvector  $|\mathbf{k}\rangle$  of transfer matrix corresponding to the largest eigenvalue  $T(\xi, \mathbf{k})$  and 'horizontal' inhomogeneities  $\xi_j = e^{v_j}$ . First we take arbitrary 'vertical' inhomogeneities  $\tau_j = e^{\beta_j}$ ,  $j = 1, \dots, N$ , so that the monodromy matrix  $T_a(\zeta, \mathbf{k}) = R_{a,N}(\lambda - \beta_N) \cdots R_{a,1}(\lambda - \beta_1) q^{k\sigma_a^z}$ ,  $\zeta = e^\lambda$

Density matrix fulfills reduction property

$$\text{tr}_1 \{ D_N(\xi_1, \dots, \xi_n | \mathbf{k}, \alpha) q^{\alpha \sigma_1^z} \} = \rho(\xi_1) D_N(\xi_2, \dots, \xi_n | \mathbf{k}, \alpha), \quad \rho(\zeta) := \frac{T(\zeta, \mathbf{k} + \alpha)}{T(\zeta, \mathbf{k})}$$

$$\text{tr}_n \{ D_N(\xi_1, \dots, \xi_n | \mathbf{k}, \alpha) \} = D_N(\xi_1, \dots, \xi_{n-1} | \mathbf{k}, \alpha)$$

# $Q$ -matrix, TBA and auxiliary function $\alpha$



Let us introduce  $Q$ -matrix with eigenvalues  $Q(\lambda, \kappa)$  satisfying the Baxter's  $TQ$ -relation:

$$T(\lambda, \kappa)Q(\lambda, \kappa) = q^{-\kappa}d(\lambda)Q(\lambda + \eta, \kappa) + q^{\kappa}a(\lambda)Q(\lambda - \eta, \kappa)$$

with  $a(\lambda) := \prod_{j=1}^N \sinh(\lambda - \beta_j + \eta)$  and  $d(\lambda) := \prod_{j=1}^N \sinh(\lambda - \beta_j)$ .

The right and left eigenvectors  $|\kappa\rangle$  and  $\langle \kappa + \alpha|$  can be constructed by means of the algebraic Bethe ansatz. They are parameterized by two sets  $\{\lambda\} = \{\lambda_j\}_{j=1}^{N/2}$  and  $\{\mu\} = \{\mu_j\}_{j=1}^{N/2}$  of Bethe roots

$$Q(\lambda, \kappa) = \prod_{j=1}^{N/2} \sinh(\lambda - \lambda_j), \quad Q(\lambda, \kappa + \alpha) = \prod_{j=1}^{N/2} \sinh(\lambda - \mu_j)$$

which are special solutions to the Bethe ansatz equations

$$\alpha(\lambda, \kappa) := q^{-2\kappa} \frac{d(\lambda)}{a(\lambda)} \frac{Q(\lambda + \eta, \kappa)}{Q(\lambda - \eta, \kappa)} = -1, \quad \alpha(\lambda, \kappa + \alpha) = -1$$

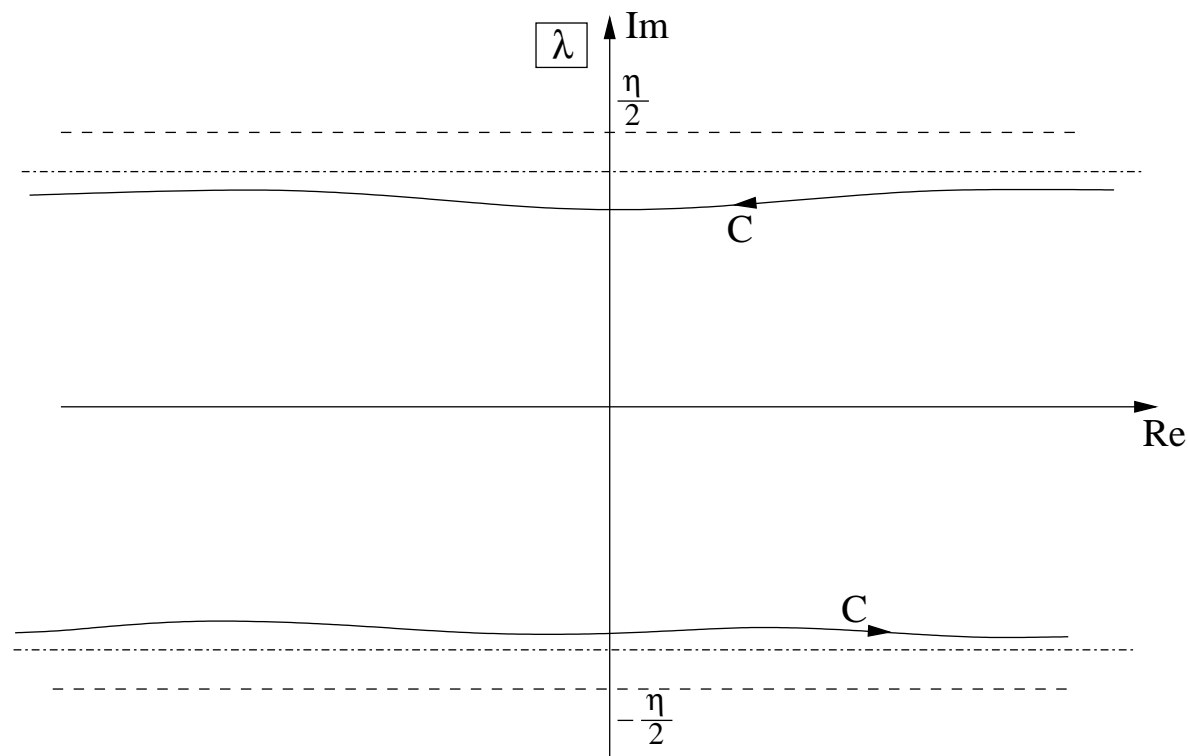
# Q-matrix, TBA and auxiliary function $\alpha$



One derives the TBA-equation:  $\ln(\alpha(\lambda, \kappa)) =$

$$(N - 2\kappa)\eta + \sum_{j=1}^N \ln \left[ \frac{\text{sh}(\lambda - \beta_j)}{\text{sh}(\lambda - \beta_j + \eta)} \right] - \int_C \frac{d\mu}{2\pi i} K(\lambda - \mu) \ln(1 + \alpha(\mu, \kappa))$$

with  $K(\lambda) = \text{cth}(\lambda - \eta) - \text{cth}(\lambda + \eta)$  and the integration contour





One can introduce temperature via alternating inhomogeneities

$$\beta_j = \begin{cases} \beta_{2j-1} = \eta - \frac{\beta}{N} \\ \beta_{2j} = \frac{\beta}{N} \end{cases}, \quad j = 1, \dots, N/2$$

with the inverse temperature  $\beta = T^{-1}$  and then taking the Trotter limit  $N \rightarrow \infty$

$$\ln(\alpha(\lambda, \kappa)) = -2\kappa\eta - \frac{\text{sh}(\eta)e(\lambda)}{T} - \int_{\mathcal{C}} \frac{d\mu}{2\pi i} K(\lambda - \mu) \ln(1 + \alpha(\mu, \kappa))$$

with the 'bare energy':  $e(\lambda) = \text{cth}(\lambda) - \text{cth}(\lambda + \eta)$

The function  $\rho$  is given through the solution of the above equation

$$\rho(\zeta) = q^\alpha \exp \left\{ \int_{\mathcal{C}} \frac{d\mu}{2\pi i} e(\mu - \lambda) \ln \left[ \frac{1 + \alpha(\mu, \kappa + \alpha)}{1 + \alpha(\mu, \kappa)} \right] \right\}, \quad \zeta = e^\lambda$$

Analytical property:  $\alpha$  has essential singularity at  $\lambda = 0$  in contour  $\mathcal{C}$  as well as at  $\lambda = \pm\eta$  outside the contour and  $\rho$  – only outside the contour at  $\zeta = q^{\pm 1}$ .

# The multiple integral representation of the density matrix



One can adapt the derivation of the multiple integrals developed by Göhmann, Klümper and Seel to the case with  $\alpha$  and obtain the following expression

$$D_{N \varepsilon_1 \dots \varepsilon_n}^{\varepsilon'_1 \dots \varepsilon'_n}(\xi_1, \dots, \xi_n) = \left[ \prod_{j=1}^p \int_{\mathcal{C}} dm(\lambda_j) F_{\ell_j}^+(\lambda_j) \right] \left[ \prod_{j=p+1}^n \int_{\mathcal{C}} d\bar{m}(\lambda_j) F_{\ell_j}^-(\lambda_j) \right] \cdot \frac{\det_{j,k=1,\dots,n} [-G(\lambda_j, \mathbf{v}_k)]}{\prod_{1 \leq j < k \leq n} \text{sh}(\lambda_j - \lambda_k - \eta) \text{sh}(\mathbf{v}_k - \mathbf{v}_j)}$$

where  $dm(\lambda) = \frac{d\lambda}{2\pi i \rho(\zeta)(1 + \alpha(\lambda, \kappa))}$ ,  $d\bar{m}(\lambda) = \alpha(\lambda, \kappa) dm(\lambda)$ ,

$$F_{\ell_j}^{\pm}(\lambda) = \prod_{k=1}^{\ell_j-1} \text{sh}(\lambda - \mathbf{v}_k) \prod_{k=\ell_j+1}^n \text{sh}(\lambda - \mathbf{v}_k \mp \eta), \quad \ell_j = \begin{cases} \varepsilon_j^+ & j = 1, \dots, p \\ \varepsilon_{n-j+1}^- & j = p+1, \dots, n \end{cases}$$

with  $\varepsilon_j^+$  the  $j$ th plus in the sequence  $(\varepsilon_j)_{j=1}^n$ ,  $\varepsilon_j^-$  the  $j$ th minus sign in the sequence  $(\varepsilon'_j)_{j=1}^n$  and  $p$  the number of plus signs in  $(\varepsilon_j)_{j=1}^n$ .

# The multiple integral representation of the density matrix



The function  $G$  is a solution to the linear integral equation

$$G(\lambda, \nu) = q^{-\alpha} \text{cth}(\lambda - \nu - \eta) - \rho(\xi) \text{cth}(\lambda - \nu) + \int_{\mathcal{C}} dm(\mu) K_{\alpha}(\lambda - \mu) G(\mu, \nu)$$

where  $\xi = e^{\nu}$  and the kernel:  $K_{\alpha}(\lambda) = q^{-\alpha} \text{cth}(\lambda - \eta) - q^{\alpha} \text{cth}(\lambda + \eta)$  is deformed version of the original kernel  $K$ .

**The case  $n = 1$**

There are only two non-vanishing density matrix elements. From the reduction we have

$$\begin{pmatrix} D_{+}^{+}(\xi) \\ D_{-}^{-}(\xi) \end{pmatrix} = \frac{1}{q^{\alpha} - q^{-\alpha}} \begin{pmatrix} -q^{-\alpha} & 1 \\ q^{\alpha} & -1 \end{pmatrix} \begin{pmatrix} 1 \\ \rho(\xi) \end{pmatrix}$$

When we insert it into the integral representation, we get an interesting identity:

$$\int_{\mathcal{C}} dm(\mu) G(\mu, \nu) = \frac{q^{-\alpha} - \rho(\nu)}{q^{\alpha} - q^{-\alpha}} \quad \text{and hence} \quad \lim_{\text{Re } \lambda \rightarrow \pm \infty} G(\lambda, \nu) = 0$$



- **Factorization** means certain reducibility of multiple integrals representing correlation functions to a combination of one-dimensional integrals with some coefficients of algebraic nature.

Example for XXX at  $T = 0, \alpha = 0, \kappa = 0$ : 
$$\langle \text{vac} | S_1^z S_3^z | \text{vac} \rangle = \frac{1}{4} - 4 \ln 2 + \frac{9}{4} \zeta(3)$$

Takahashi (77)

- We know that all correlation functions for the XXX and XXZ model at zero temperature factorize in this sense.  
Korepin, HB (01-02), Korepin, Smirnov, HB (03-04), Jimbo, Miwa, Smirnov, Takeyama, HB (04-08)
- We know that some particular correlation functions factorize for the XXX and XXZ model at finite temperature and  $\alpha = 0$  Göhmann, Klümper, Suzuki, HB (06-07)
- The claim is that it generally works for all correlation functions for the XXZ model with temperature, magnetic and disorder field. We continue with the  $n = 2$  case.

## Factorization of the density matrix for $n = 2$



There are six non-vanishing elements of the density matrix for  $n = 2$ : one for  $p = 0$ , four for  $p = 1$  and one for  $p = 2$ . We take the case  $p = 1$ . After substituting  $w_j = e^{2\mu_j}$  and  $\xi_j = e^{\nu_j}$ ,  $j = 1, 2$ , the corresponding integrals are all of the form

$$\mathcal{J} = \frac{1}{\xi_2^2 - \xi_1^2} \int_{\mathcal{C}} dm(\mu_1) \int_{\mathcal{C}} d\bar{m}(\mu_2) \det [G(\mu_j, \nu_k)] r(w_1, w_2)$$

where  $r(w_1, w_2) = \frac{p(w_1, w_2)}{w_1 - q^2 w_2}$ ,  $p(w_1, w_2) = c_0 w_1 w_2 + c_1 w_1 + c_2 w_2 + c_3$

The coefficients  $c_j$  are different for the four different matrix elements.

Inserting  $d\bar{m}(\mu) = \frac{d\mu}{2\pi i \rho(e^\mu)} - dm(\mu)$  here and taking into account that  $\rho(e^\mu)$  is analytic and non-zero inside  $\mathcal{C}$  we obtain

$$\begin{aligned} \mathcal{J}(\xi_2^2 - \xi_1^2) = & - \int_{\mathcal{C}} dm(\mu) \det \begin{pmatrix} G(\mu, \nu_1) & G(\mu, \nu_2) \\ r(w, \xi_1^2) & r(w, \xi_2^2) \end{pmatrix} \\ & - \int_{\mathcal{C}} dm(\mu_1) \int_{\mathcal{C}} dm(\mu_2) \det [G(\mu_j, \nu_k)] r(w_1, w_2), \quad w = e^{2\mu} \end{aligned}$$

## Factorization of the density matrix for $n = 2$



The key idea is then to find rational functions  $f(w)$  and  $g(w)$  such that

$$r(w_1, w_2) - r(w_2, w_1) = f(w_1) - f(w_2) + g(w_1)K_\alpha(\mu_1 - \mu_2) - g(w_2)K_\alpha(\mu_2 - \mu_1)$$

and taking residues at  $w_1 = q^2 w_2$  we obtain a difference equation for  $g$

$$g(q^2 w)y^{-1} - g(w)y = \frac{p(q^2 w, w)}{2q^2 w}, \quad y := q^\alpha$$

which can be solved:  $g(w) = g_+ w + g_0 + \frac{g_-}{w}$

with the coefficients  $g_+ = \frac{c_0 y}{2(q^2 - y^2)}$ ,  $g_- = \frac{c_3 y}{2(1 - q^2 y^2)}$ ,  $g_0 = \frac{(c_1 + q^{-2} c_2) y}{2(1 - y^2)}$

and hence  $f(w) = (y - y^{-1}) \left( g_+ w - \frac{g_-}{w} \right)$ . Collecting all terms, we arrive at

$$\mathcal{J} = \frac{g(\xi_2^2)\Psi(\xi_2, \xi_1) - g(\xi_1^2)\Psi(\xi_1, \xi_2)}{\xi_2^2 - \xi_1^2} + \frac{(c_1 - q^{-2} c_2)(\rho(\xi_1) - \rho(\xi_2))}{2(\xi_2^2 - \xi_1^2)(y - y^{-1})} + \frac{(y^{-1} - \rho(\xi_1))(y - \rho(\xi_2))f(\xi_2^2) - (y^{-1} - \rho(\xi_2))(y - \rho(\xi_1))f(\xi_1^2)}{(\xi_2^2 - \xi_1^2)(y - y^{-1})^2},$$

where  $\Psi(\xi_1, \xi_2) = \int_{\mathcal{C}} dm(\mu) G(\mu, \mathbf{v}_2) (q^\alpha \text{cth}(\mu - \mathbf{v}_1 - \eta) - \rho(\xi_1) \text{cth}(\mu - \mathbf{v}_1))$

# From fermionic operators to fermionic basis



The factorized elements of the density matrix look complicated since the standard basis might be not so convenient. We suggested another basis based on fermionic operators. **Jimbo, Miwa, Smirnov, Takeyama, HB (08–09)**

Consider a space  $\mathcal{W}^{(\alpha)} = \bigoplus_{s=-\infty}^{\infty} \mathcal{W}_{\alpha-s,s}$  where  $\mathcal{W}_{\alpha-s,s}$  is the space of quasi-local operators of spin  $s$  with tail  $\alpha - s$ .

The creation operators  $\mathbf{t}^*(\zeta)$ ,  $\mathbf{b}^*(\zeta)$ ,  $\mathbf{c}^*(\zeta)$  and annihilation operators  $\mathbf{b}(\zeta)$ ,  $\mathbf{c}(\zeta)$  act on  $\mathcal{W}^{(\alpha)}$ . They are one-parameter families of operators of the form

$$\mathbf{t}^*(\zeta) = \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{t}_p^*,$$

$$\mathbf{b}^*(\zeta) = \zeta^{\alpha+2} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{b}_p^*, \quad \mathbf{c}^*(\zeta) = \zeta^{-\alpha-2} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{c}_p^*,$$

$$\mathbf{b}(\zeta) = \zeta^{-\alpha} \sum_{p=0}^{\infty} (\zeta^2 - 1)^{-p} \mathbf{b}_p, \quad \mathbf{c}(\zeta) = \zeta^{\alpha} \sum_{p=0}^{\infty} (\zeta^2 - 1)^{-p} \mathbf{c}_p$$

# From fermionic operators to fermionic basis



The operator  $\mathbf{t}_p^*$  commutes with all other operators, the operators  $\mathbf{b}_p, \mathbf{c}_p, \mathbf{b}_p^*, \mathbf{c}_p^*$  with  $p > 0$  are fermions with canonical anti-commutation relations. Fourier modes have block structure:

$$\mathbf{t}_p^* : \mathcal{W}_{\alpha-s,s} \rightarrow \mathcal{W}_{\alpha-s,s}$$

$$\mathbf{b}_p^*, \mathbf{c}_p : \mathcal{W}_{\alpha-s+1,s-1} \rightarrow \mathcal{W}_{\alpha-s,s}, \quad \mathbf{c}_p^*, \mathbf{b}_p : \mathcal{W}_{\alpha-s-1,s+1} \rightarrow \mathcal{W}_{\alpha-s,s}.$$

We call the operator  $q^{\alpha \sum_{j=-\infty}^0 \sigma_j^z}$  a primary field. The annihilation operators kill the primary field. The creation operators acting on the primary field produce different quasi-local operators. We proved that the operators

$$\tau^m \mathbf{t}_{p_1}^* \cdots \mathbf{t}_{p_j}^* \mathbf{b}_{q_1}^* \cdots \mathbf{b}_{q_k}^* \mathbf{c}_{r_1}^* \cdots \mathbf{c}_{r_k}^* \left( q^{\alpha \sum_{j=-\infty}^0 \sigma_j^z} \right), \quad \tau = \mathbf{t}_1^*/2, \quad m \in \mathbb{Z}, \quad j, k \in \mathbb{Z}_{\geq 0},$$

$$p_1 \geq \cdots \geq p_j \geq 2, \quad q_1 > \cdots > q_k \geq 1, \quad r_1 > \cdots > r_k \geq 1$$

constitute a basis of  $\mathcal{W}_{\alpha,0}$ . Length of quasi-local operator is not bigger than  $\sum p_m + \sum q_m + \sum r_m$ . Clearly, this description of  $\mathcal{W}_{\alpha,0}$  reminds that of CFT.

# From fermionic operators to fermionic basis



Important theorem was proved by **Jimbo, Miwa and Smirnov (09)**

$$Z^{\kappa} \{ \mathbf{t}^*(\zeta)(X) \} = 2\rho(\zeta) Z^{\kappa} \{ X \},$$

$$Z^{\kappa} \{ \mathbf{b}^*(\zeta)(X) \} = \frac{1}{2\pi i} \oint_{\Gamma} \omega(\zeta, \xi) Z^{\kappa} \{ \mathbf{c}(\xi)(X) \} \frac{d\xi^2}{\xi^2},$$

$$Z^{\kappa} \{ \mathbf{c}^*(\zeta)(X) \} = -\frac{1}{2\pi i} \oint_{\Gamma} \omega(\xi, \zeta) Z^{\kappa} \{ \mathbf{b}(\xi)(X) \} \frac{d\xi^2}{\xi^2}$$

where the contour  $\Gamma$  goes around  $\xi^2 = 1$  and  $\omega$  is some explicit function. These formulae allow one to explicitly calculate

$$\begin{aligned} & Z^{\kappa} \{ \mathbf{t}^*(\zeta_1^0) \cdots \mathbf{t}^*(\zeta_p^0) \mathbf{b}^*(\zeta_1^+) \cdots \mathbf{b}^*(\zeta_q^+) \mathbf{c}^*(\zeta_q^-) \cdots \mathbf{c}^*(\zeta_1^-) (q^{\alpha \sum_{j=-\infty}^0 \sigma_j^z}) \} = \\ & = \prod_{i=1}^p 2\rho(\zeta_i^0) \det \left| \omega(\zeta_i^+, \zeta_j^-) \right|_{i,j=1, \dots, q} \quad \text{generating function for series in } \zeta^2 - 1 \end{aligned}$$



The above theorem states that

- any correlation function corresponding to some quasi-local operator  $\mathcal{O}$  can be reduced to a combination of two transcendental functions  $\rho$  and  $\omega$ . They depend on temperature and magnetic field in contrast to the basis which is pure algebraic.
- the basis is independent of inhomogeneities in the Matsubara direction
- one can take a finite Trotter number  $N$  and arbitrary inhomogeneities there and find  $\rho$  and  $\omega$ . In this case  $\omega$  was found to be related to the deformed abelian integrals. It satisfies the so-called normalization condition

$$\int_{\Gamma_m} T(\zeta, \kappa) \omega(\zeta, \xi) Q^-(\zeta, \kappa + \alpha) Q^+(\zeta, \kappa) \varphi(\zeta) \frac{d\zeta^2}{\zeta^2}, \quad m = 0, \dots, N$$

where  $\Gamma_0$  encircles 0 and  $\Gamma_m$  encircle two poles  $\zeta^2 = \tau_m^2$  and  $\zeta^2 = \tau_m^2/q^2$ ,

$Q^\pm$  are eigenvalues of  $Q$ -matrices corresponding to two linear independent

solutions of the  $TQ$ -relation and  $\varphi(\zeta) := \prod_{m=1}^N \frac{1}{(\zeta^2/\tau_m^2 - 1)(\zeta^2/\tau_m^2/q^2 - 1)}$ .



The above formulation of the  $\omega$ -function is very beautiful but it is not quite clear how to take the Trotter limit  $N \rightarrow \infty$  after specifying the inhomogeneities.

The further **Theorem** stays:

F. Göhmann, HB (09)

The function  $\omega$  has an alternative representation in terms of the TBA-functions

$$\begin{aligned}\omega_{\text{rat}}(\xi_1, \xi_2) &= 2(\xi_1/\xi_2)^\alpha \Psi(\xi_1, \xi_2) - \Delta \psi(\xi) + 2(\rho(\xi_1) - \rho(\xi_2))\psi(\xi) \\ \omega(\xi_1, \xi_2) &= \omega_{\text{rat}}(\xi_1, \xi_2) + \bar{D}_\zeta \bar{D}_\xi \Delta_\zeta^{-1} \psi(\zeta/\xi)\end{aligned}$$

where  $\psi(\xi) = \frac{\xi^\alpha(\xi^2+1)}{2(\xi^2-1)}$ ,  $\Delta_\xi$  is the difference operator whose action on a function  $f$  is defined by  $\Delta_\xi f(\xi) = f(q\xi) - f(q^{-1}\xi)$  and

$$\bar{D}_\zeta g(\zeta) = g(q\zeta) + g(q^{-1}\zeta) - 2\rho(\zeta)g(\zeta)$$



*Sketch of the proof:*

First we return to the case of finite  $N$  and arbitrary inhomogeneities. We use the fact that the function  $(\zeta/\xi)^{-\alpha}\omega(\zeta, \xi)$  is a rational function of  $\zeta^2$  of the form  $P(\zeta^2)/Q(\zeta^2)$  with polynomials  $P, Q$  of degree at most  $N + 2$ . The zeros of  $Q$  are the  $N$  zeros of the transfer matrix eigenvalue  $T(\zeta, \kappa)$  and two additional zeros at  $q^{\pm 2}\xi^2$ . Then we prove that the function  $\omega$  satisfies the normalization

$$\text{condition: } \omega(\zeta, \xi)|_{\zeta=\tau_j} + \rho(\tau_j)\omega(\zeta, \xi)|_{\zeta=q^{-1}\tau_j} = 0$$

corresponding to the contours  $\Gamma_m$  with  $m = 1, \dots, N$  and

$$\lim_{\zeta \rightarrow 0} (\zeta/\xi)^{-\alpha}\omega(\zeta, \xi) = \frac{2q^{-\kappa}}{q^{\kappa} + q^{-\kappa}} \left[ \rho(\xi) - q^{-\alpha} + (q^{\alpha} - q^{-\alpha}) \int_{\mathcal{C}} dm(\mu) G(\mu, \nu) \right] = 0$$

corresponding to the contour  $\Gamma_0$ . All together we have to provide  $N + 3$  relations. We have  $N + 1$  normalization conditions. Two more relations come from considering the residues at  $\zeta^2 = q^{\pm 2}\xi^2$ .

# The exponential form – preliminary remarks



The formula by Jimbo, Miwa and Smirnov makes possible the calculation of arbitrary correlation functions because of completeness of the fermionic basis. It also proves the factorization of the correlation functions. Sometimes it is still more convenient to avoid the creation operators and to have an explicit formula for the correlation functions in the standard basis. We know that such a formula exists in the zero temperature case:

$$\frac{\langle \text{vac} | q^{\alpha \sum_{j=-\infty}^0 \sigma_j^z} \mathcal{O} | \text{vac} \rangle}{\langle \text{vac} | q^{\alpha \sum_{j=-\infty}^0 \sigma_j^z} | \text{vac} \rangle} = \mathbf{tr}^{\alpha} \left\{ \exp(\Omega) \left( q^{\alpha \sum_{j=-\infty}^0 \sigma_j^z} \mathcal{O} \right) \right\}$$

where

$$\Omega = \int \frac{d\zeta_1^2}{2\pi i \zeta_1^2} \int \frac{d\zeta_2^2}{2\pi i \zeta_2^2} \omega(\zeta_1 / \zeta_2, \alpha) \mathbf{b}(\zeta_1) \mathbf{c}(\zeta_2)$$

with some explicit function  $\omega$  and

$$\mathbf{tr}^{\alpha}(X) = \cdots \text{tr}_1^{\alpha} \text{tr}_2^{\alpha} \text{tr}_3^{\alpha} \cdots (X), \quad \text{tr}^{\alpha}(x) = \text{tr} \left( q^{-\frac{1}{2} \alpha \sigma^3} x \right) / \text{tr} \left( q^{-\frac{1}{2} \alpha \sigma^3} \right), \quad x \in \text{End}(\mathbb{C}^2)$$



**Conjecture:** Similar formula does exist for the temperature case also

$$Z^{\kappa} \left\{ q^{\alpha \sum_{j=-\infty}^0 \sigma_j^z} \mathcal{O} \right\} = \mathbf{tr}^{\alpha} \left\{ \exp(\Omega) \left( q^{\alpha \sum_{j=-\infty}^0 \sigma_j^z} \mathcal{O} \right) \right\}$$

where

$$\Omega = \Omega_1 + \Omega_2$$

$\Omega_1$  has a similar structure as in  $T = 0$  case

$$\Omega_1 = \int \frac{d\zeta_1^2}{2\pi i \zeta_1^2} \int \frac{d\zeta_2^2}{2\pi i \zeta_2^2} (\omega_0(\zeta_1/\zeta_2|\alpha) - \omega_{\text{rat}}(\zeta_1, \zeta_2|\kappa, \alpha)) \mathbf{b}(\zeta_1) \mathbf{c}(\zeta_2)$$

$$\omega_0(\zeta|\alpha) = - \left( \frac{1 - q^{\alpha}}{1 + q^{\alpha}} \right)^2 \Delta_{\zeta} \Psi(\zeta)$$

The second part should be of the form:

$$\Omega_2 = \int \frac{d\zeta^2}{2\pi i \zeta^2} \log(\rho(\zeta)) \mathbf{t}(\zeta)$$

where the operator  $\mathbf{t}$  is yet to be determined.

# The exponential form – preliminary remarks



Let us list some of the most important expected properties of the operator  $\mathbf{t}$ . First, we expect that like  $\mathbf{t}^*(\zeta)$  the operator  $\mathbf{t}(\zeta)$  is block diagonal,

$$\mathbf{t}(\zeta) : \mathcal{W}_{\alpha,s} \rightarrow \mathcal{W}_{\alpha,s}$$

We will deal below mostly with the sector  $s = 0$ . Then we expect  $\mathbf{t}(\zeta)$  to have simple poles at  $\zeta = \xi_j$ . Let us define:  $\mathbf{t}_j = \text{res}_{\zeta=\xi_j} \mathbf{t}(\zeta) \frac{d\zeta^2}{\zeta^2}$ ,  $\mathbf{t}_j^* = \mathbf{t}^*(\xi_j)$

In contrast to  $\mathbf{t}_j$  the operator  $\mathbf{t}_j^*$  is well defined only if it acts on the states  $X_{[m,n]}$  with  $m \leq n < j$ . Let us denote  $\mathbf{t}_{[m,n]}(\zeta)$  and respectively  $\mathbf{t}_{j[m,n]}$  the operators defined on the interval  $[m,n]$  with  $m \leq j \leq n$ . We also expect the  $R$ -matrix symmetry

$$\mathbf{s}_i \mathbf{t}_{[m,n]}(\zeta) = \mathbf{t}_{[m,n]}(\zeta) \mathbf{s}_i \quad \text{for } m \leq i < n, \quad \mathbf{s}_i = K_{i,i+1} \check{\mathbb{R}}_{i,i+1}(\xi_i/\xi_{i+1})$$

$$\check{\mathbb{R}}_{i,i+1}(\xi_i/\xi_{i+1})(X) = \check{R}_{i,i+1}(\xi_i/\xi_{i+1}) X \check{R}_{i,i+1}(\xi_i/\xi_{i+1})^{-1}$$

$K_{i,j}$  stands for the transposition of arguments  $\xi_i$  and  $\xi_j$ .



The further properties are:

- Commutation relations:  $[\mathbf{t}_j, \mathbf{t}_k]_- = [\mathbf{t}_j, \mathbf{b}(\zeta_1)\mathbf{c}(\zeta_2)]_- = 0$
- Projector property:  $\mathbf{t}_j^2 = \mathbf{t}_j$
- Relations with  $\mathbf{t}^*$ :
 
$$\mathbf{t}_j \mathbf{t}_k^* = \mathbf{t}_j^* \mathbf{t}_k \quad \text{for } j \neq k$$

$$\mathbf{t}_j \mathbf{t}_j^* = \mathbf{t}_j^*, \quad \mathbf{t}_j^* \mathbf{t}_j = 0$$
- Reduction properties:

$$\mathbf{t}_{1[1,n]}(q^{\alpha\sigma_1^z} X_{[2,n]}) = q^{\alpha\sigma_1^z} X_{[2,n]}$$

$$\mathbf{t}_{j[1,n]}(q^{\alpha\sigma_1^z} X_{[2,n]}) = q^{\alpha\sigma_1^z} \mathbf{t}_{j[2,n]}(X_{[2,n]}) \quad \text{for } 1 < j \leq n$$

$$\mathbf{t}_{j[1,n]}(X_{[1,n-1]}) = \mathbf{t}_{j[1,n-1]}(X_{[1,n-1]}) \quad \text{for } 1 \leq j < n$$

$$\mathbf{t}_{n[1,n]}(X_{[1,n-1]}) = 0.$$

# The exponential form – preliminary remarks



Let us comment on these relations.

First of all the commutation relations lead to the factorization of the exponential

$$\exp(\Omega) = \exp(\Omega_1 + \Omega_2) = \exp(\Omega_1) \exp(\Omega_2).$$

As we know the operator  $\Omega_1$  becomes nilpotent when it acts on states of finite length. In contrast, the operator  $\Omega_2$  is not nilpotent, but due to the commutation relations and the projector property

$$\exp(\Omega_2) \left( q^{\alpha \sum_{j=-\infty}^0 \sigma_j^z} X_{[1,n]} \right) = \prod_{j=1}^n (1 - \mathbf{t}_{j[1,n]} + \rho_j \mathbf{t}_{j[1,n]}) \left( X_{[1,n]} \right) q^{\alpha \sum_{j=-\infty}^0 \sigma_j^z}$$

where  $\rho_j = \rho(\xi_j)$ .

The reduction properties look standard except for the first one. It is easy to see that we need all of them in order to have the reduction property of the density matrix. For the moment we know explicit expression of  $\mathbf{t}$  only for some particular cases.



- The main purpose of my talk was to show that the factorization works for the temperature case of the  $XXZ$  model at magnetic and disorder field despite of all complications coming from the TBA-approach. I think it could be a hint that factorization of correlation functions might be a fundamental property of integrable models.
- In case of the  $XXZ$  model we could find rather suitable fermionic basis for which the determinant formula is valid. It looks similar to the Slater-determinant of the free fermion model.
- The whole non-trivial information about the structure of the Matsubara space is hidden in two transcendental functions  $\rho$  and  $\omega$ .
- Now we know that one can take the scaling limit of these functions and get the results for the one-point correlator of CFT.

joint paper with M.Jimbo, T.Miwa and F. Smirnov (09)