Asymptotic behavior of correlation functions from Bethe ansatz: the XXZ chain

Jean-Michel Maillet

CNRS & ENS Lyon, France

Collaborators : *N. Kitanine, K.K. Kozlowski, N.A Slavnov, V. Terras*.

- "Riemann-Hilbert approach to a generalised sine kernel and applications" `Comm. Math. Phys. 291, 691-761 (2009)`
- "Algebraic Bethe ansatz approach to the asymptotic behavior of correlation functions" `J. Stat. Mech. P04003 (2009)`
- "On the thermodynamic limit of form factors in the massless XXZ Heisenberg chain" `J. Math. Phys. 50, 095209 (2009)`

The spin-1/2 XXZ Heisenberg chain

The *XXZ* spin-1/2 Heisenberg chain **in a magnetic field** is a quantum interacting model defined on a one-dimensional lattice with M sites, with Hamiltonian,

$$H_{XXZ} = \sum_{m=1}^{M} \{\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta(\sigma_m^z \sigma_{m+1}^z - 1)\} - \mathbf{h} \sum_{m=1}^{M} \sigma_m^z$$

Quantum space of states : $\mathcal{H} = \otimes_{m=1}^{M} \mathcal{H}_m$, $\mathcal{H}_m \sim \mathbb{C}^2$, $\dim \mathcal{H} = 2^M$.

$\sigma_m^{x,y,z}$: local spin operators (in the spin-$\frac{1}{2}$ representation) at site m
They act as the corresponding Pauli matrices in the space $\mathcal{H}_m$ and as the identity operator elsewhere.

- periodic boundary conditions
- disordered regime, $|\Delta| < 1$ and $h < h_c$

Correlation functions of Heisenberg chain

- **Exact results**
  - Free fermion point $\Delta = 0$: Lieb, Shultz, Mattis, Wu, McCoy, Sato, Jimbo, Miwa . . .
  - From 1984: Izergin, Korepin . . . (first attempts using Bethe ansatz for general Δ)
  - General Δ: multiple integral representations (for building blocks)
    $\star$ 1992-96 Jimbo, Miwa . . . $\longrightarrow$ from q-vertex op. and qKZ eq.
    $\star$ 1999 Kitanine, Maillet, Terras $\longrightarrow$ from Algebraic Bethe Ansatz
  - Several developments since 2000: Kitanine, Maillet, Slavnov, Terras; Boos, Korepin, Smirnov; Boos, Jimbo, Miwa, Smirnov, Takeyama; Göhmann, Klümper, Seel; Caux, Hagemans, Maillet . . .
- **Asymptotic results** $\langle \sigma_1^\alpha \sigma_m^\beta \rangle \underset{m\to\infty}{\sim}$?
  - Luttinger liquid approximation / C.F.T. and finite size effects
    Luther and Peschel, Haldane, Cardy, Affleck, ... Lukyanov, ...

$\hookrightarrow$ **Asymptotic behavior from exact results ?**

Algebraic Bethe ansatz and correlation functions

Compute $\langle \psi_g \,|\, \sigma_1^{\alpha}\, \sigma_{m+1}^{\beta} \,|\, \psi_g \rangle$?

1. **Diagonalise the Hamiltonian using ABA**
   $\longrightarrow$ key point : Yang-Baxter algebra $A(\lambda)$, $B(\lambda)$, $C(\lambda)$, $D(\lambda)$
   $\longrightarrow$ $|\,\psi_g\,\rangle = B(\lambda_1)\dots B(\lambda_N)|\,0\,\rangle$ with $\mathcal{Y}(\lambda_j;\{\lambda\}) = 0$ (Bethe eq.)
2. **Act with local operators on eigenstates**
   $\longrightarrow$ solve the quantum inverse problem (1999):
   $$\sigma_j^{\alpha_j} = f_j^{\alpha_j}(A,B,C,D) = \prod(A,B,C,D)$$
   $\longrightarrow$ use Yang-Baxter commutation relations
3. **Compute the resulting scalar products** (determinant representation)
   $\longrightarrow$ determinant representation for form factors of the finite chain
   $\longrightarrow$ elementary building blocks of correlation functions as multiple integrals in the thermodynamic limit (2000)
4. **Two-point function:** sum up elementary blocks or form factors
   $\longrightarrow$ **Master equation representation** for the finite chain (2005)
5. **Asymptotic analysis of the two-point function:**
   $\longrightarrow$ Series expansion of the Master equation at the thermodynamic limit
   $\longrightarrow$ Asymptotic analysis of the series (2008 - 2009)

The σ^z correlation functions : generating function

$$Q^{\kappa}_{1,m} = \prod_{n=1}^{m} \left(\frac{1+\kappa}{2} + \frac{1-\kappa}{2} \cdot \sigma^z_n \right)$$

Equivalently $Q^{\kappa}_{1,m} = e^{\beta \mathcal{Q}_{1m}}$ with $\mathcal{Q}_{1m} = \frac{1}{2} \sum_{n=1}^{m} (1 - \sigma^z_n)$ and $\kappa = e^{\beta}$.

$$\langle \sigma^z_1 \sigma^z_{m+1} \rangle = 2\, D^2_m\, \partial^2_{\kappa} \left. \langle Q^{\kappa}_{1,m} \rangle \right|_{\kappa=1} + 2\langle \sigma^z \rangle - 1 \quad \text{with } D^2_m u_m = u_{m+1} + u_{m-1} - 2u_m$$

- Inverse problem: $Q^{\kappa}_{1,m} = \mathcal{T}_{\kappa}(0)^m \cdot \mathcal{T}_{\kappa=1}(0)^{-m}$

  with $\mathcal{T}_{\kappa}(\nu) = A(\nu) + \kappa D(\nu)$ twisted transfer matrix

  $\rightsquigarrow$ $\mathcal{T}_{\kappa}(\nu) \, | \, \psi_{\kappa}(\{\mu\}) \, \rangle = \tau_{\kappa}(\nu | \{\mu\}) \, | \, \psi_{\kappa}(\{\mu\}) \, \rangle$

  if $\{\mu\}$ is solution of the κ-twisted Bethe equations $\mathcal{Y}_{\kappa}(\mu_j | \{\mu\}) = 0$

  $\rightsquigarrow$ $\left. \frac{\mathrm{d}}{\mathrm{d}\mu} \log \mathcal{T}_{\kappa=1}(\mu) \right|_{\mu=0} \propto H_{XXZ}$

- Act with $\mathcal{T}_{\kappa}(0)^m \cdot \mathcal{T}_{\kappa=1}(0)^{-m}$ on $| \, \psi_g \, \rangle$ or

  Sum over κ-deformed form-factors

  $\Longrightarrow$ Master equation

The σ^z correlation functions : generating function

$$Q^{\kappa}_{1,m} = \prod_{n=1}^{m} \left(\frac{1+\kappa}{2} + \frac{1-\kappa}{2} \cdot \sigma^z_n \right)$$

Equivalently $Q^{\kappa}_{1,m} = e^{\beta \mathcal{Q}_{1m}}$ with $\mathcal{Q}_{1m} = \frac{1}{2} \sum_{n=1}^{m} (1 - \sigma^z_n)$ and $\kappa = e^{\beta}$.

$$\langle \sigma^z_1 \sigma^z_{m+1} \rangle = 2\, D^2_m\, \partial^2_\kappa \, \langle Q^{\kappa}_{1,m} \rangle \Big|_{\kappa=1} + 2 \langle \sigma^z \rangle - 1 \quad \text{with } D^2_m u_m = u_{m+1} + u_{m-1} - 2u_m$$

- Inverse problem: $Q^{\kappa}_{1,m} = \mathcal{T}_\kappa(0)^m \cdot \mathcal{T}_{\kappa=1}(0)^{-m}$

  with $\mathcal{T}_\kappa(\nu) = A(\nu) + \kappa D(\nu)$ twisted transfer matrix

  $\rightsquigarrow$ $\mathcal{T}_\kappa(\nu) \,|\, \psi_\kappa(\{\mu\}) \,\rangle = \tau_\kappa(\nu|\{\mu\}) \,|\, \psi_\kappa(\{\mu\}) \,\rangle$

  if $\{\mu\}$ is solution of the κ-twisted Bethe equations $\mathcal{Y}_\kappa(\mu_j|\{\mu\}) = 0$

  $\rightsquigarrow$ $\frac{\mathrm{d}}{\mathrm{d}\mu} \log \mathcal{T}_{\kappa=1}(\mu) \Big|_{\mu=0} \propto H_{XXZ}$

- Act with $\mathcal{T}_\kappa(0)^m \cdot \mathcal{T}_{\kappa=1}(0)^{-m}$ on $|\, \psi_g \,\rangle$ or

  Sum over κ-deformed form-factors

  $\Longrightarrow$ Master equation

Master equation for the finite chain

$$\langle Q^{\kappa}_{1,m}\rangle = \frac{\langle\, \psi_g\,|\mathcal{T}_{\kappa}(0)^m \cdot \mathcal{T}_{\kappa=1}(0)^{-m}|\,\psi_g\,\rangle}{\langle \psi_g|\psi_g\rangle}$$

→ $\langle Q^{\kappa}_{1,m}\rangle$ polynomial in κ

→ for κ small enough the spectrum of $\mathcal{T}_{\kappa}$ is simple, well separated from the one at $\kappa = 1$, described by κ-twisted Bethe equations $\mathcal{Y}_{\kappa}(\mu_j|\{\mu\}) = 0$, and κ-twisted Bethe states $|\,\psi_{\kappa}(\{\mu\})\,\rangle$ form a complete basis

$$\langle Q^{\kappa}_{1,m}\rangle = \sum_{\substack{\{\mu\} \text{ solutions of} \\ \text{twisted Bethe eq.}}} \frac{\langle \psi_g|\psi_{\kappa}(\{\mu\})\rangle \cdot \langle \psi_{\kappa}(\{\mu\})|\psi_g\rangle}{\langle \psi_{\kappa}(\{\mu\})|\psi_{\kappa}(\{\mu\})\rangle \cdot \langle \psi_g|\psi_g\rangle} \cdot \frac{\tau_{\kappa}(0|\{\mu\})^m}{\tau(0|\{\lambda\})^m}$$

→ can be rewritten as a multiple contour integral around the solutions $\{\mu\}$ of the κ-twisted Bethe equations, with the product of these twisted Bethe equations $\mathcal{Y}_{\kappa}(z_j|\{z\})$ in denominator (the norm squared of these states is related to the Jacobian of the κ-twisted Bethe equations).

$$\langle Q^{\kappa}_{1,m}\rangle = \frac{1}{N!} \oint_{\substack{\Gamma(\{\mu\}) \\ \text{solutions of} \\ \text{twisted Bethe eq.}}} \frac{\mathrm{d}^N z}{(2\pi i)^N} \prod_{j=1}^{N} \left[e^{im[p_0(z_j) - p_0(\lambda_j)]}\, \frac{d(z_j)}{d(\lambda_j)} \right] \frac{\left[\det_N \Omega_{\kappa}(\{z\},\{\lambda\}|\{z\})\right]^2}{\prod_{j=1}^{N} \mathcal{Y}_{\kappa}(z_j|\{z\}) \cdot \det_N \frac{\partial \mathcal{Y}(\lambda_j|\{\lambda\})}{\partial \lambda_k}}$$

Master equation for the finite chain

$$\langle Q^{\kappa}_{1,m}\rangle = \frac{\langle\, \psi_g \,|\mathcal{T}_{\kappa}(0)^m \cdot \mathcal{T}_{\kappa=1}(0)^{-m}|\, \psi_g \,\rangle}{\langle \psi_g|\psi_g\rangle}$$

$\longrightarrow$ $\langle Q^{\kappa}_{1,m}\rangle$ polynomial in κ

$\longrightarrow$ for κ small enough the spectrum of $\mathcal{T}_{\kappa}$ is simple, well separated from the one at $\kappa = 1$, described by κ-twisted Bethe equations $\mathcal{Y}_{\kappa}(\mu_j|\{\mu\}) = 0$, and κ-twisted Bethe states $|\,\psi_{\kappa}(\{\mu\})\,\rangle$ form a complete basis

$$\langle Q^{\kappa}_{1,m}\rangle = \sum_{\substack{\{\mu\} \text{ solutions of} \\ \text{twisted Bethe eq.}}} \frac{\langle \psi_g|\psi_{\kappa}(\{\mu\})\rangle \cdot \langle \psi_{\kappa}(\{\mu\})|\psi_g\rangle}{\langle \psi_{\kappa}(\{\mu\})|\psi_{\kappa}(\{\mu\})\rangle \cdot \langle \psi_g|\psi_g\rangle} \cdot \frac{\tau_{\kappa}(0|\{\mu\})^m}{\tau(0|\{\lambda\})^m}$$

$\longrightarrow$ can be rewritten as a multiple contour integral around the solutions $\{\mu\}$ of the κ-twisted Bethe equations, with the product of these twisted Bethe equations $\mathcal{Y}_{\kappa}(z_j|\{z\})$ in denominator (the norm squared of these states is related to the Jacobian of the κ-twisted Bethe equations).

$$\langle Q^{\kappa}_{1,m}\rangle = \frac{1}{N!} \oint_{\substack{\Gamma(\{\mu\}) \\ \text{solutions of} \\ \text{twisted Bethe eq.}}} \frac{\mathrm{d}^N z}{(2\pi i)^N} \prod_{j=1}^{N} \left[e^{im[p_0(z_j)-p_0(\lambda_j)]} \frac{d(z_j)}{d(\lambda_j)} \right] \frac{\left[\det_N \Omega_{\kappa}(\{z\},\{\lambda\}|\{z\})\right]^2}{\prod_{j=1}^{N} \mathcal{Y}_{\kappa}(z_j|\{z\}) \cdot \det_N \frac{\partial \mathcal{Y}(\lambda_j|\{\lambda\})}{\partial \lambda_k}}$$

Master equation for the finite chain

$$\langle Q^{\kappa}_{1,m}\rangle = \frac{\langle\, \psi_g\, |\mathcal{T}_{\kappa}(0)^m \cdot \mathcal{T}_{\kappa=1}(0)^{-m}|\, \psi_g\,\rangle}{\langle \psi_g | \psi_g\rangle}$$

$\longrightarrow$ $\langle Q^{\kappa}_{1,m}\rangle$ polynomial in κ

$\longrightarrow$ for κ small enough the spectrum of $\mathcal{T}_{\kappa}$ is simple, well separated from the one at $\kappa = 1$, described by κ-twisted Bethe equations $\mathcal{Y}_{\kappa}(\mu_j|\{\mu\}) = 0$, and κ-twisted Bethe states $|\,\psi_{\kappa}(\{\mu\})\,\rangle$ form a complete basis

$$\langle Q^{\kappa}_{1,m}\rangle = \sum_{\substack{\{\mu\}\text{ solutions of}\\ \text{twisted Bethe eq.}}} \frac{\langle \psi_g|\psi_{\kappa}(\{\mu\})\rangle \cdot \langle \psi_{\kappa}(\{\mu\})|\psi_g\rangle}{\langle \psi_{\kappa}(\{\mu\})|\psi_{\kappa}(\{\mu\})\rangle \cdot \langle \psi_g|\psi_g\rangle} \cdot \frac{\tau_{\kappa}(0|\{\mu\})^m}{\tau(0|\{\lambda\})^m}$$

$\longrightarrow$ can be rewritten as a multiple contour integral around the solutions $\{\mu\}$ of the κ-twisted Bethe equations, with the product of these twisted Bethe equations $\mathcal{Y}_{\kappa}(z_j|\{z\})$ in denominator (the norm squared of these states is related to the Jacobian of the κ-twisted Bethe equations).

$$\langle Q^{\kappa}_{1,m}\rangle = \frac{1}{N!} \oint_{\substack{\Gamma(\{\mu\})\\ \text{solutions of}\\ \text{twisted Bethe eq.}}} \frac{d^N z}{(2\pi i)^N} \prod_{j=1}^{N} \left[e^{im[p_0(z_j)-p_0(\lambda_j)]} \frac{d(z_j)}{d(\lambda_j)} \right] \frac{\left[\det_N \Omega_{\kappa}(\{z\},\{\lambda\}|\{z\})\right]^2}{\prod_{j=1}^{N} \mathcal{Y}_{\kappa}(z_j|\{z\}) \cdot \det_N \frac{\partial \mathcal{Y}(\lambda_j|\{\lambda\})}{\partial \lambda_k}}$$

Master equation for the finite chain

$$\langle Q^{\kappa}_{1,m}\rangle = \frac{\langle\, \psi_g\, |\mathcal{T}_\kappa(0)^m \cdot \mathcal{T}_{\kappa=1}(0)^{-m}|\, \psi_g\,\rangle}{\langle \psi_g|\psi_g\rangle}$$

$\rightarrow$ $\langle Q^{\kappa}_{1,m}\rangle$ polynomial in κ

$\rightarrow$ for κ small enough the spectrum of $\mathcal{T}_\kappa$ is simple, well separated from the one at $\kappa = 1$, described by κ-twisted Bethe equations $\mathcal{Y}_\kappa(\mu_j|\{\mu\}) = 0$, and κ-twisted Bethe states $|\,\psi_\kappa(\{\mu\})\,\rangle$ form a complete basis

$$\langle Q^{\kappa}_{1,m}\rangle = \sum_{\substack{\{\mu\}\text{ solutions of}\\ \text{twisted Bethe eq.}}} \frac{\langle\psi_g|\psi_\kappa(\{\mu\})\rangle \cdot \langle\psi_\kappa(\{\mu\})|\psi_g\rangle}{\langle\psi_\kappa(\{\mu\})|\psi_\kappa(\{\mu\})\rangle \cdot \langle\psi_g|\psi_g\rangle} \cdot \frac{\tau_\kappa(0|\{\mu\})^m}{\tau(0|\{\lambda\})^m}$$

$\rightarrow$ can be rewritten as a multiple contour integral around the solutions $\{\mu\}$ of the κ-twisted Bethe equations, with the product of these twisted Bethe equations $\mathcal{Y}_\kappa(z_j|\{z\})$ in denominator (the norm squared of these states is related to the Jacobian of the κ-twisted Bethe equations).

$$\langle Q^{\kappa}_{1,m}\rangle = \frac{1}{N!} \oint_{\substack{\Gamma(\{\mu\})\\ \text{solutions of}\\ \text{twisted Bethe eq.}}} \frac{\mathrm{d}^N z}{(2\pi i)^N} \prod_{j=1}^{N} \left[e^{im[p_0(z_j)-p_0(\lambda_j)]} \frac{d(z_j)}{d(\lambda_j)} \right] \frac{\left[\det_N \Omega_\kappa(\{z\},\{\lambda\}|\{z\})\right]^2}{\prod_{j=1}^{N} \mathcal{Y}_\kappa(z_j|\{z\}) \cdot \det_N \frac{\partial \mathcal{Y}(\lambda_j|\{\lambda\})}{\partial \lambda_k}}$$

Remark 1. $\exists$ 2 ways to write scalar product between Bethe state $|\,\psi(\{\lambda\})\,\rangle$ and κ-twisted Bethe state $|\,\psi_\kappa(\{\mu\})\,\rangle$ (cf. N. Slavnov's determinant formula):

- in term of $\det_N \Omega_\kappa(\{\mu\},\{\lambda\}|\{\mu\})$ if we use the κ-deformed Bethe eq. for $\{\mu\}$
- in term of $\det_N \Omega(\{\lambda\},\{\mu\}|\{\lambda\})$ if we use the (standard) Bethe eq. for $\{\lambda\}$

$\rightsquigarrow$ **different forms for master equation:**

- with simple poles at $\{\lambda\}$ (parameters for the ground state) and $\{\xi\}$ (inhomogeneity parameters) $+$ poles at $\{\mu\}$ solutions of κ-twisted Bethe equations (initial form obtained from multiple integrals)
- with **double poles** at $\{\lambda\}$ (parameters for the ground state) $+$ poles at $\{\mu\}$ solutions of κ-twisted Bethe equations (form we use here)

Remark 2. Integrals can be computed by residues inside or outside the contour:

$$\langle Q^{\kappa}_{1,m}\rangle = \frac{(-1)^N}{N!}\oint_{\Gamma(\{\lambda\})}\frac{d^N z}{(2\pi i)^N}\prod_{j=1}^{N}\left[e^{im[p_0(z_j)-p_0(\lambda_j)]}\frac{d(z_j)}{d(\lambda_j)}\right]\frac{\Big[\det_N \Omega_\kappa(\{z\},\{\lambda\}|\{z\})\Big]^2}{\prod_{j=1}^{N}\mathcal{Y}_\kappa(z_j|\{z\})\,\det_N \frac{\partial \mathcal{Y}(\lambda_j|\{\lambda\})}{\partial \lambda_k}}$$

$\rightsquigarrow$ contour around double poles at the parameters $\lambda_1,\ldots,\lambda_N$ describing the ground state

Remark 1. $\exists$ 2 ways to write scalar product between Bethe state $|\,\psi(\{\lambda\})\,\rangle$ and κ-twisted Bethe state $|\,\psi_\kappa(\{\mu\})\,\rangle$ (cf. N. Slavnov's determinant formula):

- in term of $\det_N \Omega_\kappa(\{\mu\},\{\lambda\}|\{\mu\})$ if we use the κ-deformed Bethe eq. for $\{\mu\}$
- in term of $\det_N \Omega(\{\lambda\},\{\mu\}|\{\lambda\})$ if we use the (standard) Bethe eq. for $\{\lambda\}$

$\rightsquigarrow$ **different forms for master equation:**

- with simple poles at $\{\lambda\}$ (parameters for the ground state) and $\{\xi\}$ (inhomogeneity parameters) + poles at $\{\mu\}$ solutions of κ-twisted Bethe equations (initial form obtained from multiple integrals)
- with **double poles** at $\{\lambda\}$ (parameters for the ground state) + poles at $\{\mu\}$ solutions of κ-twisted Bethe equations (form we use here)

Remark 2. Integrals can be computed by residues inside or outside the contour:

$$\langle Q^{\kappa}_{1,m}\rangle = \frac{(-1)^N}{N!}\oint_{\Gamma(\{\lambda\})}\frac{\mathrm{d}^N z}{(2\pi i)^N}\prod_{j=1}^{N}\left[e^{im[p_0(z_j)-p_0(\lambda_j)]}\,\frac{d(z_j)}{d(\lambda_j)}\right]\frac{\Big[\det_N \Omega_\kappa(\{z\},\{\lambda\}|\{z\})\Big]^2}{\prod_{j=1}^{N}\mathcal{Y}_\kappa(z_j|\{z\})\,\det_N \frac{\partial \mathcal{Y}(\lambda_j|\{\lambda\})}{\partial\lambda_k}}$$

$\rightsquigarrow$ contour around double poles at the parameters $\lambda_1,\ldots,\lambda_N$ describing the ground state

The series expansion: thermodynamic limit

- ⇝ **Single out the poles "$z_j = \lambda_k$":**

$$\det_N \Omega_\kappa(\{z\},\{\lambda\}|\{z\}) = \det_{\mathbf{N}} \left[\frac{1}{\sinh(z_j-\lambda_k)}\right] \cdot \det_N T_\kappa(\{z\},\{\lambda\}|\{z\})$$

- ⇝ Reorganize and expand determinants → **Series expansion**
- ⇝ **Thermodynamic limit** $(N, M \to \infty,\ N/M \to D,\ \{\lambda\} \to \rho(\lambda)$ on $[-q,q])$:

$$\langle e^{\beta \mathcal{Q}_{1m}} \rangle = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{-q}^{q} \frac{d^n\lambda}{(2i\pi)^n} \oint_{\Gamma([-q,q])} \frac{d^n z}{(2i\pi)^n} \prod_{\ell=1}^{n} \frac{e^{im[p_0(z_\ell)-p_0(\lambda_\ell)]}}{\sinh(z_\ell-\lambda_\ell)} \times \det_n\left[\frac{1}{\sinh(z_k-\lambda_j)}\right] \cdot \mathcal{F}_n^{(\kappa)}\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix}$$

with $\mathcal{F}_n^{(\kappa)}$ symmetric in $\{\lambda\}$ and $\{z\}$ + satisfy reduction properties at "$z_j = \lambda_k$"

- ⋆ if $\mathcal{F}_n^{(\kappa)} = \prod_{i=1}^{n}[\varphi(\lambda_i)\, e^{g(z_i)}]$ decoupled → **Fredholm determinant** :

$$\langle e^{\beta \mathcal{Q}_{1m}} \rangle = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{-q}^{q} \frac{d^n\lambda}{(2i\pi)^n} \det_n[V(\lambda_j,\lambda_k)]$$

$$V(\lambda,\mu) = \varphi(\lambda) e^{g(\lambda)} \frac{\sin\left\{\frac{m}{2}[p_0(\lambda)-p_0(\mu)] - \frac{i}{2}[g(\lambda)-g(\mu)]\right\}}{\pi \sinh(\lambda-\mu)}$$

- ⋆ $\mathcal{F}_n^{(\kappa)}$ not decoupled ?

The series expansion: thermodynamic limit

⇝ **Single out the poles** "$z_j = \lambda_k$":

$$\det_N \Omega_\kappa(\{z\},\{\lambda\}|\{z\}) = \det_{\mathbf{N}} \left[\frac{1}{\sinh(z_j - \lambda_k)}\right] \cdot \det_N T_\kappa(\{z\},\{\lambda\}|\{z\})$$

⇝ Reorganize and expand determinants → **Series expansion**

⇝ **Thermodynamic limit** $(N, M \to \infty,\ N/M \to D,\ \{\lambda\} \to \rho(\lambda)$ on $[-q,q])$:

$$\langle e^{\beta \mathcal{Q}_{1m}} \rangle = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{-q}^{q} \frac{\mathrm{d}^n\lambda}{(2i\pi)^n} \oint_{\Gamma([-q,q])} \frac{\mathrm{d}^n z}{(2i\pi)^n} \prod_{\ell=1}^{n} \frac{e^{im[p_0(z_\ell)-p_0(\lambda_\ell)]}}{\sinh(z_\ell - \lambda_\ell)} \times \det_n \left[\frac{1}{\sinh(z_k - \lambda_j)}\right] \cdot \mathcal{F}_n^{(\kappa)} \begin{pmatrix} \{\lambda\} \\ \{z\} \end{pmatrix}$$

with $\mathcal{F}_n^{(\kappa)}$ symmetric in $\{\lambda\}$ and $\{z\}$ + satisfy reduction properties at "$z_j = \lambda_k$"

⋆ if $\mathcal{F}_n^{(\kappa)} = \prod_{i=1}^{n}[\varphi(\lambda_i)\, e^{g(z_i)}]$ decoupled → **Fredholm determinant** :

$$\langle e^{\beta \mathcal{Q}_{1m}} \rangle = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{-q}^{q} \frac{\mathrm{d}^n\lambda}{(2i\pi)^n} \det_n[V(\lambda_j,\lambda_k)]$$

$$V(\lambda,\mu) = \varphi(\lambda) e^{g(\lambda)} \frac{\sin\left\{\frac{m}{2}[p_0(\lambda) - p_0(\mu)] - \frac{i}{2}[g(\lambda) - g(\mu)]\right\}}{\pi \sinh(\lambda - \mu)}$$

⋆ $\mathcal{F}_n^{(\kappa)}$ not decoupled ?

The series expansion: decomposition into cycle integrals

→ **Cycle expansion of the determinant**

$$\frac{1}{n!}\int d^n\lambda \int d^n z\, \mathcal{G}_n\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix} \det_n[g(z_j,\lambda_k)]$$

$$= \sum_{\substack{\ell_1,\dots,\ell_n=0 \\ \Sigma k\ell_k=n}} C(n|\{\ell\}) \int d^n\lambda \int d^n z \underbrace{\mathcal{G}_n\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix}}_{\substack{\text{symmetric} \\ \text{in } \{\lambda\} \text{ and } \{z\}}} \prod_{s=1}^{n}\prod_{p=1}^{\ell_s}\prod_{j=1}^{s} g(z_{s,p,j},\lambda_{s,p,j})$$

cycles labelled by (s,p), variables labelled by (s,p,j), $1\le p\le \ell_s$, $1\le j\le s$
s=length of a cycle ℓ_s=number of cycles of length s

$$\langle e^{\beta \mathcal{Q}_m}\rangle = \sum_{n=0}^{+\infty} \sum_{\substack{\ell_1,\dots,\ell_n=0 \\ \Sigma k\ell_k=n}} C(n|\{\ell\}) \left\{\prod_{s=1}^{n}\prod_{p=1}^{\ell_s} \mathfrak{I}_{(s,p)}\right\} [\mathcal{F}_n^{(\kappa)}]$$

Each $\mathfrak{I}_{s,p}$ integrates over the variables $\lambda_{s,p,j}$ and $z_{s,p,j}$ with $1\le j\le s$.

$$\mathfrak{I}_s[\mathcal{G}_s] = \oint_{\Gamma([-q,q])} \frac{d^s z}{(2i\pi)^s} \int_{-q}^{q} \frac{d^s\lambda}{(2i\pi)^s}\, \mathcal{G}_s\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix} \prod_{j=1}^{s} \frac{\exp\{im[p_0(\lambda_j)-p_0(z_j)]\}}{\sinh(z_j-\lambda_j)\sinh(z_j-\lambda_{j+1})}$$

The series expansion: decomposition into cycle integrals

→ **Cycle expansion of the determinant**

$$\frac{1}{n!}\int \mathrm{d}^n\lambda \int \mathrm{d}^n z\, \mathcal{G}_n\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix} \det_n[g(z_j,\lambda_k)]$$

$$= \sum_{\substack{\ell_1,\dots,\ell_n=0\\ \Sigma k\ell_k=n}} C(n|\{\ell\}) \int \mathrm{d}^n\lambda \int \mathrm{d}^n z\; \underbrace{\mathcal{G}_n\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix}}_{\substack{\text{symmetric}\\ \text{in } \{\lambda\} \text{ and } \{z\}}} \prod_{s=1}^{n}\prod_{p=1}^{\ell_s}\prod_{j=1}^{s} g(z_{s,p,j},\lambda_{s,p,j})$$

cycles labelled by (s,p), variables labelled by (s,p,j), $1\le p\le \ell_s$, $1\le j\le s$
s=length of a cycle $\qquad$ ℓ_s=number of cycles of length s

$$\langle e^{\beta \mathcal{Q}_m}\rangle = \sum_{n=0}^{+\infty} \sum_{\substack{\ell_1,\dots,\ell_n=0\\ \Sigma k\ell_k=n}} C(n|\{\ell\}) \left\{ \prod_{s=1}^{n}\prod_{p=1}^{\ell_s} \mathfrak{I}_{(s,p)} \right\} [\mathcal{F}_n^{(\kappa)}]$$

Each $\mathfrak{I}_{s,p}$ integrates over the variables $\lambda_{s,p,j}$ and $z_{s,p,j}$ with $1\le j\le s$.

$$\mathfrak{I}_s[\mathcal{G}_s] = \oint_{\Gamma([-q,q])} \frac{\mathrm{d}^s z}{(2i\pi)^s} \int_{-q}^{q} \frac{\mathrm{d}^s\lambda}{(2i\pi)^s}\, \mathcal{G}_s\begin{pmatrix}\{\lambda\}\\ \{z\}\end{pmatrix} \prod_{j=1}^{s} \frac{\exp\{im\,[p_0(\lambda_j)-p_0(z_j)]\}}{\sinh(z_j-\lambda_j)\sinh(z_j-\lambda_{j+1})}$$

Cycle integrals and generalized sine kernel

$$\mathfrak{I}_s[\mathcal{G}_s] = \oint_{\Gamma([-q,q])} \frac{\mathrm{d}^s z}{(2i\pi)^s} \int_{-q}^{q} \frac{\mathrm{d}^s \lambda}{(2i\pi)^n} \, \mathcal{G}_s \begin{pmatrix} \{\lambda\} \\ \{z\} \end{pmatrix} \prod_{j=1}^{s} \frac{\exp\{im\,[p_0(\lambda_j) - p_0(z_j)]\}}{\sinh(z_j - \lambda_j)\sinh(z_j - \lambda_{j+1})}$$

with $\mathcal{G}_s$ symmetric separately in $\{\lambda\}$ and in $\{z\}$

- if $\mathcal{G}_s \begin{pmatrix} \{\lambda\} \\ \{z\} \end{pmatrix} = \prod_{i=1}^{s} \left[\varphi(\lambda_i)\, e^{g(z_i)}\right]$ then $\mathfrak{I}_s[\mathcal{G}_s]$ can be obtained in terms of the **Fredholm determinant** of a **generalized sine kernel** :

$$\mathfrak{I}^{(s)}[\mathcal{G}_s] = \int_{-q}^{q} \mathrm{d}^n\lambda \prod_{j=1}^{s} V^{(\varphi,g)}(\lambda_j, \lambda_{j+1}) = \frac{(-1)^{s-1}}{(s-1)!} \frac{\partial^s}{\partial \gamma^s} \log\det[I + \gamma V^{(\varphi,g)}]\Big|_{\gamma=0}$$

$$V^{(\varphi,g)}(\lambda,\mu) = F(\lambda) \frac{\sin\left\{\frac{m}{2}[p_0(\lambda) - p_0(\mu)] - \frac{i}{2}[g(\lambda) - g(\mu)]\right\}}{\pi \sinh(\lambda - \mu)}$$

with $F(\lambda) = \varphi(\lambda) e^{g(\lambda)}$

- **density theorem** in the general case: $\mathcal{G}_s \begin{pmatrix} \{\lambda\} \\ \{z\} \end{pmatrix} = \sum_{\ell=1}^{\infty} \prod_{i=1}^{s} \left[\varphi_\ell(\lambda_i)\, e^{g_\ell(z_i)}\right]$

$\hookrightarrow$ Analyze the asymptotic behavior of the generalized sine kernel and apply the density procedure to get the asymptotic of $\mathfrak{I}_s$

Asymptotic behavior of cycle integrals

- **Asymptotics of the generalized sine-kernel**
  → **Matrix Riemann-Hilbert Problems** (generalization of the procedure of Deift, Its, Zhou (1997) for the sine-kernel)
- **Application to cycle integrals**
  → take the n^{th} γ-derivative
  → specialize to $V^{(\varphi,g)}$
  → apply the density procedure (corrections remain corrections)

$$\mathfrak{I}_s[\mathcal{G}_s] = H_s[\mathcal{G}_s] + D_s[\mathcal{G}_s] + O_s[\mathcal{G}_s]$$

$$H_s[\mathcal{G}_s] = \frac{1}{2\pi i}\int_{-q}^{q} \mathrm{d}\lambda \Big\{ imp_0'(\lambda) - b_s \log\big(m \sinh(2q) p_0'(\lambda)\big) \times[\delta(\lambda+q)+\delta(\lambda-q)]\Big\} \mathcal{G}_s\begin{pmatrix}\lambda,\dots,\lambda\\ \lambda,\dots,\lambda\end{pmatrix} + C[\mathcal{G}_s]$$

$$D_s[\mathcal{G}_s] = \int_{-q}^{q} \frac{\mathrm{d}\lambda}{2i\pi}\, \partial_\epsilon\, \mathcal{G}_s\begin{pmatrix}\lambda,\lambda,\dots,\lambda\\ \lambda+\epsilon,\lambda,\dots,\lambda\end{pmatrix}\Bigg|_{\epsilon=0} + \cdots \text{ (derivative)}$$

$O_s[\mathcal{G}_s]$ = terms of order o(1) (contains oscillating contributions $e^{irmp_0(\pm q)}$)

Asymptotic summation of the series

$$\langle e^{\beta \mathcal{Q}_{1m}} \rangle = \sum_{n=0}^{+\infty} \sum_{\substack{\ell_1,\dots,\ell_n=0 \\ \sum k\ell_k = n}} C(n|\{\ell\}) \prod_{s=1}^{n} \prod_{p=1}^{\ell_s} \left\{ H_{s,p} + D_{s,p} + O_{s,p} \right\} \left[\mathcal{F}_n^{(\kappa)} \right]$$

Sum up successively (use binomial formula) H_s, then D_s, then O_s
$+$ use the **reduction properties of $\mathcal{F}_n^{(\kappa)}$ at $z_j = \lambda_k$**

- $\longrightarrow$ the series of H_s exponentiates
- $\longrightarrow$ the series of successive actions of D_s is a continuous generalization of the multiple Lagrange series : its sum is expressed in terms of a solution of an integral equation
- $\longrightarrow$ sum-up O_s pertubatively

Asymptotic summation of the series

The series we have to sum up is in fact a functional version of the standard Lagrange series of the type

$$G_0^{(h)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d}{d\epsilon} + h \right)^n \left(F(\epsilon)\phi^n(\epsilon) \right) \Big|_{\epsilon=0}$$

where $F(z)$ and $\phi(z)$ are some functions holomorphic in a vicinity of the origin. If the series is convergent, then it can be summed up in terms of the solution of the equation $z - \phi(z) = 0$ and the sum is given by

$$G_0^{(h)} = \frac{F(z)\, e^{hz}}{1 - \phi'(z)}$$

In the correlation function case, z becomes a function and $\Phi(z)$ an integral operator acting on this function; hence the summation is given as the value of some functional in a point determined by an integral equation.

Results

Generating function

$$\langle e^{\beta \mathcal{Q}_{1m}} \rangle = \underbrace{G^{(0)}(\beta, m)[1 + o(1)]}_{\text{non-oscillating terms}} + \underbrace{\sum_{\sigma=\pm} G^{(0)}(\beta + 2i\pi\sigma, m)[1 + o(1)]}_{\text{oscillating terms}}$$

$$G^{(0)}(\beta, m) = C(\beta)\, e^{m\beta D}\, m^{\frac{\beta^2}{2\pi^2} Z(q)^2}$$

- $Z(\lambda)$ is the dressed charge $\quad Z(\lambda) + \int_{-q}^{q} \frac{\mathrm{d}\mu}{2\pi} K(\lambda - \mu)\, Z(\mu) = 1$
- D is the average density $\quad D = \int_{-q}^{q} \rho(\mu)\mathrm{d}\mu = \frac{1 - \langle \sigma^z \rangle}{2} = \frac{p_F}{\pi}$
- The coefficient $C(\beta)$ is given as the ratio of four Fredholm determinants.
- sub-leading oscillating terms restore the $2\pi i$-periodicity in β

2-point function

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = (2D - 1)^2 - \frac{2Z(q)^2}{\pi^2 m^2} + 2|F_{\sigma^z}|^2 \cdot \frac{\cos(2mp_F)}{m^{2Z(q)^2}} + o\left(\frac{1}{m^2}, \frac{1}{m^{2Z(q)^2}}\right)$$

Results

Generating function

$$\langle e^{\beta \mathcal{Q}_{1m}} \rangle = \underbrace{G^{(0)}(\beta, m)[1 + o(1)]}_{\text{non-oscillating terms}} + \underbrace{\sum_{\sigma=\pm} G^{(0)}(\beta + 2i\pi\sigma, m)[1 + o(1)]}_{\text{oscillating terms}}$$

$$G^{(0)}(\beta, m) = C(\beta)\, e^{m\beta D}\, m^{\frac{\beta^2}{2\pi^2} Z(q)^2}$$

- $Z(\lambda)$ is the dressed charge $\quad Z(\lambda) + \int_{-q}^{q} \frac{\mathrm{d}\mu}{2\pi} K(\lambda - \mu)\, Z(\mu) = 1$
- D is the average density $\quad D = \int_{-q}^{q} \rho(\mu)\mathrm{d}\mu = \frac{1 - \langle \sigma^z \rangle}{2} = \frac{p_F}{\pi}$
- The coefficient $C(\beta)$ is given as the ratio of four Fredholm determinants.
- sub-leading oscillating terms restore the $2\pi i$-periodicity in β

2-point function

$$\langle \sigma_1^z \sigma_{m+1}^z \rangle = (2D - 1)^2 - \frac{2Z(q)^2}{\pi^2 m^2} + 2|F_{\sigma^z}|^2 \cdot \frac{\cos(2mp_F)}{m^{2Z(q)^2}} + o\left(\frac{1}{m^2}, \frac{1}{m^{2Z(q)^2}}\right)$$

Form factors

The umklapp form factor

$$\lim_{N,M\to\infty}\left(\frac{M}{2\pi}\right)^{2\mathcal{Z}^2}\frac{|\langle\psi(\{\mu\})|\sigma^z|\psi(\{\lambda\})\rangle|^2}{\|\psi(\{\mu\})\|^2\cdot\|\psi(\{\lambda\})\|^2}=|F_\sigma|^2.$$

with

$$2\mathcal{Z}^2=Z(q)^2+Z(-q)^2$$

- $\{\lambda\}$ are the Bethe parameters of the ground state
- $\{\mu\}$ are the Bethe parameters for the excited state with one particle and one hole on opposite sides of the Fermi boundary (umklapp type excitation).

$\hookrightarrow$ It means that the above form factor behaves as $M^{-2\mathcal{Z}^2}$ for M large. Hence, the exponent $2\mathcal{Z}^2$ governing the modulus squared of the form factor power-law decrease in terms of the size M of the chain is exactly equal to the exponent for the power-law behavior of the corresponding oscillating term in the correlation function. Can be generalized to n-particle form factors.

Form factors

The umklapp form factor

$$\lim_{N,M\to\infty}\left(\frac{M}{2\pi}\right)^{2\mathcal{Z}^2}\frac{|\langle\psi(\{\mu\})|\sigma^z|\psi(\{\lambda\})\rangle|^2}{\|\psi(\{\mu\})\|^2\cdot\|\psi(\{\lambda\})\|^2}=|F_\sigma|^2.$$

with

$$2\mathcal{Z}^2 = Z(q)^2 + Z(-q)^2$$

- $\{\lambda\}$ are the Bethe parameters of the ground state
- $\{\mu\}$ are the Bethe parameters for the excited state with one particle and one hole on opposite sides of the Fermi boundary (umklapp type excitation).

$\hookrightarrow$ It means that the above form factor behaves as $M^{-2\mathcal{Z}^2}$ for M large. Hence, the exponent $2\mathcal{Z}^2$ governing the modulus squared of the form factor power-law decrease in terms of the size M of the chain is exactly equal to the exponent for the power-law behavior of the corresponding oscillating term in the correlation function. Can be generalized to n-particle form factors.

Some open problems . . .

- Limit $h = 0$: check of Lukyanov's predictions for the amplitudes
- Other correlation functions $\langle \sigma_1^+ \sigma_{m+1}^- \rangle$, ...
- Other models (already done: density-density correlation function of the quantum one-dimensional Bose gas)
- Simpler method using form factors as suggested by the result
    - NLS : long distance and long time asymptotics
    - XXZ : work in progress
    - Sine-Gordon?