Hidden fermionic structure of XXZ-model.

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Quantum affine algebra $U_q'(\widehat{\mathfrak{sl}}_2)$.

The algebra $U_Q'(\widehat{\mathfrak{sl}}_2)$is generated by e_i, f_i, h_i $(i = 0, 1)$. We consider the case of central charge equal to zero: $h_1 = -h_0 \equiv h$. Two Borel subalgebras $U_q(\mathfrak{b}^+)$ and $U_q(\mathfrak{b}^-)$ are generated respectively by e_i, h and f_i, h. We have the commutation relations:

$$[e_i, f_j] = \delta_{i,j} \frac{t_i - t_i^{-1}}{q - q^{-1}} ,$$

where $t_i = q^{h_i}$. The deformed Serre relations are

$$e_i^3 e_j + (q^2 + q^{-2} + 1)(e_i^2 e_j e_i - e_i e_j e_i^2) - e_j e_i^3 = 0 ,$$
$$f_i^3 f_j + (q^2 + q^{-2} + 1)(f_i^2 f_j f_i - f_i f_j f_i^2) - f_j f_i^3 = 0$$

The comultiplication is

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta(t_i) = t_i \otimes t_i ,$$

The universal R-matrix intertwining Δ and $\Delta' = \Delta \circ \mathrm{Perm}$ is complicated.
It can be written as follows:

$$\mathcal{R} = \overline{\mathcal{R}} q^{-\frac{h \otimes h}{2}} \,,$$

$$\overline{\mathcal{R}} = 1 - (q - q^{-1}) \sum_{i=0}^{1} e_i \otimes f_j + \cdots \ \in \ U_q(\mathfrak{b}^{+}) \otimes U_q(\mathfrak{b}^{-}) \,,$$

Representations.

1. Evaluation representation

$$ev_\zeta(e_0) = \zeta F, \quad ev_\zeta(e_1) = \zeta E, \quad ev_\zeta(f_0) = \zeta^{-1} E, \quad ev_\zeta(f_1) = \zeta^{-1} F, \quad ev_\zeta(h) = H \,,$$

2. Finite-dimensional representations $\pi^{(2s)}$ correspond to $(2s + 1)$-dimensional representation of E, F, H.

The first L-operator:

$$L(\zeta) = (ev_{\zeta_1} \otimes \pi^{(1)}_{\zeta_2})(\mathcal{R}) = \tau(\zeta) L^\circ(\zeta)\,, \quad \zeta = \zeta_1/\zeta_2\,,$$

$$L^\circ(\zeta) = \begin{pmatrix} 1-\zeta^2 q^{H+1} & -(q-q^{-1})\zeta F \\ -(q-q^{-1})\zeta E & 1-\zeta^2 q^{-H+1} \end{pmatrix} t_0^{\sigma^3/2}\,,$$

3. q-oscillator representation of $U_q\mathfrak{b}^+$. The q-oscillator algebra Osc is an associative algebra with generators $\mathbf{a}, \mathbf{a}^*, q^D$, and defining relations

$$\begin{aligned} &q^D \mathbf{a}\, q^{-D} = q^{-1}\mathbf{a}, \quad q^D\, \mathbf{a}^* q^{-D} = q\, \mathbf{a}^*, \\ &\mathbf{a}\,\mathbf{a}^* = 1 - q^{2D+2}, \quad \mathbf{a}^*\mathbf{a} = 1 - q^{2D}\,. \end{aligned}$$

Representations of Osc relevant to us are $\rho^\pm : Osc \to \mathrm{End}(W^\pm)$

$$\begin{aligned} &W^+ = \oplus_{k\geq 0}\mathbb{C}|k\rangle, \quad W^- = \oplus_{k<0}\mathbb{C}|k\rangle, \\ &q^D|k\rangle = q^k|k\rangle,\ \mathbf{a}|k\rangle = (1-q^{2k})|k-1\rangle,\ \mathbf{a}^*|k\rangle = (1-\delta_{k,-1})|k+1\rangle\,. \end{aligned}$$

There is a homomorphism of algebras $o_\zeta : U_q\mathfrak{b}^+ \to Osc$ given by

$$o_\zeta(e_0) = \frac{\zeta}{q-q^{-1}}\mathbf{a}, \quad o_\zeta(e_1) = \frac{\zeta}{q-q^{-1}}\mathbf{a}^*, \quad o_\zeta(t_0) = q^{-2D}, \quad o_\zeta(t_1) = q^{2D}\,.$$

We define representations $o^\pm_\zeta : U_q\mathfrak{b}^+ \to \mathrm{End}(W^\pm)$ by

$$o^+_\zeta = \rho^+ \circ o_\zeta, \quad o^-_\zeta = \rho^- \circ o_\zeta \circ \iota\,,$$

where ι denotes the involution $e_i \to e_{1-i}$, $t_i \to t_{1-i}$ of $U_q\mathfrak{b}^+$. L-operator

$$(o^\pm_\zeta \otimes \pi_\xi)\mathcal{R} = \sigma(\zeta/\xi)\cdot L^\circ_{Aj}{}^\pm(\zeta/\xi)\,,$$

$$L^\circ_{A,j}(\zeta) = \begin{pmatrix} 1-\zeta^2 q^{2D_A+2} & -\zeta\mathbf{a}_A \\ -\zeta\mathbf{a}^*_A & 1 \end{pmatrix}_j \begin{pmatrix} q^{-D_A} & 0 \\ 0 & q^{D_A} \end{pmatrix}_j\,.$$

Trace functional $\mathrm{Tr}(q^{2\alpha D}\cdot) : Osc \to \mathbb{C}(q^{\alpha})$.

For each $x \in Osc$ and $y \in \mathbb{C}$, the ordinary trace $\pm\mathrm{Tr}_{W^{\pm}}(y^{D}x)$ on $W^{\pm}$ is well-defined for sufficiently small $|y|^{\pm 1}$, and gives the same rational function $g_x(y)$ in y. By definition, $\mathrm{Tr}(q^{2\alpha D}x)$ means $g_x(q^{2\alpha}) \in \mathbb{C}(q^{\alpha})$. Notice that $\mathrm{Tr}(q^{2\alpha D}\cdot)$ is a purely algebraic operation characterized as the unique linear map with the properties

$$\mathrm{Tr}(q^{2\alpha D}XY) = \mathrm{Tr}(q^{2\alpha D}q^{2\alpha d(X)}YX) \quad (X, Y \in Osc, q^{D}Xq^{-D} = q^{d(X)}X),$$

$$\mathrm{Tr}(q^{2\alpha D}q^{mD}) = \frac{1}{1 - q^{2\alpha+m}} \quad (m \in \mathbb{Z}).$$

Fundamental fusion relation.

By direct computation one gets

$$\begin{aligned}
L_{\{a,A\},j}(\zeta) &= (F_{a,A})^{-1}L_{a,j}(\zeta)L_{A,j}(\zeta)F_{a,A}\\
&= \begin{pmatrix} 1 & 0 \\ \frac{q-q^{-1}}{\zeta-\zeta^{-1}}\,\sigma_j^+ & 1 \end{pmatrix}_a \begin{pmatrix} L_{A,j}(q\zeta)q^{-\sigma_j^3/2} & 0 \\ 0 & L_{A,j}(q^{-1}\zeta)q^{\sigma_j^3/2} \end{pmatrix}_a ,
\end{aligned}$$

where $F_{a,A} = 1 - \mathbf{a}_A\sigma_a^+$.
This implies, in particular, Baxter's TQ-equation.
I shall need one additional property (crossing):

$$L^0_{a,j}(\zeta)^{-1} = \frac{1}{(\zeta^2-q^2)(\zeta^2-q^{-2})}\,\sigma_j^2 L^0_{a,j}(\zeta)^{t_j}\sigma_j^2\,,$$

$$L^0_{A,j}(\zeta)^{-1} = \frac{1}{(\zeta^2-1)}\,\sigma_j^2 L^0_{A,j}(\zeta)^{t_j}\sigma_j^2\,.$$

Operators on finite lattice.

Consider $V_{[k,l]} = V_k \otimes V_{k+1} \otimes \cdots \otimes V_l$, with $V_j \simeq \mathbb{C}^2$. Let

$$T_{a,[k,l]}(\zeta) = L^{\circ}_{a,l}(\zeta) \cdots L^{\circ}_{a,k}(\zeta)\,,$$

and similarly for the q-oscillator (index A). Let $X_{[k,l]} \in \mathrm{End}(V_{[k,l]})$. Our main arm is the adjoint monodromy matrices:

$$\mathbb{T}_a(\zeta,\alpha)(X_{[k,l]}) = T_{a,[k,l]}(\zeta) q^{\alpha\sigma^3_a} X_{[k,l]} T_{a,[k,l]}(\zeta)^{-1}\,,\,,$$
$$\mathbb{T}_A(\zeta,\alpha)(X_{[k,l]}) = T_{A,[k,l]}(\zeta) q^{2\alpha D_A} X_{[k,l]} T_{A,[k,l]}(\zeta)^{-1}\,.$$

Notice that $\mathbb{T}_a(\zeta,\alpha)(X_{[k,l]})$ has high order poles at $\zeta^2 = q^{\pm 2}$, $\mathbb{T}_A(\zeta,\alpha)(X_{[k,l]})$ has high order pole at $\zeta^2 = 1$, Define

$$\mathbb{S}(X_{[k,l]}) = [S_{[k,l]}, X_{[k,l]}], \quad S_{[k,l]} = \tfrac{1}{2}\textstyle\sum_{j\in[k,l]} \sigma^3_j\,.$$

Due to the universal fusion relation we have

$$\begin{aligned}\mathbb{T}_{\{a,A\}}(\zeta,\alpha)(X_{[k,l]}) &= (F_{a,A})^{-1}\left(\mathbb{T}_a(\zeta,\alpha)\mathbb{T}_A(\zeta,\alpha)(X_{[k,l]})\right)F_{a,A}\\ &= \begin{pmatrix}\mathbb{A}_A(\zeta,\alpha)(X_{[k,l]}) & 0\\ \mathbb{C}_A(\zeta,\alpha)(X_{[k,l]}) & \mathbb{D}_A(\zeta,\alpha)(X_{[k,l]})\end{pmatrix}_a,\end{aligned}$$

where

$$\mathbb{A}_A(\zeta,\alpha)(X_{[k,l]}) = \mathbb{T}_A(q\zeta,\alpha)q^{\alpha-\mathbb{S}}(X_{[k,l]}),$$
$$\mathbb{D}_A(\zeta,\alpha)(X_{[k,l]}) = \mathbb{T}_A(q^{-1}\zeta,\alpha)q^{-\alpha+\mathbb{S}}(X_{[k,l]}).$$

Operator $\mathbf{k}(\zeta,\alpha)$.

$$\mathbf{k}(\zeta,\alpha)(X_{[k,l]}) := \mathrm{Tr}_A\left\{\mathbb{C}_A(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}}\left(q^{-2S_{[k,l]}}\,X_{[k,l]}\right)\right\}.$$

Important: $\mathbf{k}(\zeta,\alpha)(X_{[k,l]})$ has poles at $\zeta^2=1,q^{\pm 2}$.

Reduction relations.

1. Left reduction relation.
By the very construction it is quite obvious that for an operator $X_{[k,l]} = q^{2(\alpha+1)S_{[k,m-1]}} \otimes Y_{[m,l]}$ with $k < m < l$ we have

$$\mathbf{k}(\zeta,\alpha)(q^{2(\alpha+1)S_{[k,m-1]}} \otimes Y_{[m,l]}) = q^{2\alpha S_{[k,m-1]}} \otimes \mathbf{k}(\zeta,\alpha)(Y_{[m,l]})\,,$$

2. Right reduction relation.
Much less obvious thihg. Let $X_{[k,l]} = Y_{[k,m]} \otimes I_{[m+1,l]}$ then

$$\mathbf{k}(\zeta,\alpha)(Y_{[k,m]} \otimes I_{[m+1,l]}) = \mathbf{k}(\zeta,\alpha)(Y_{[k,m]}) \otimes I_{[m+1,l]} + \Delta_\zeta \mathbf{v}(\zeta,\alpha)(Y_{[k,m]} \otimes I_{[m+1,l]})\,,$$

where $\mathbf{v}$ is rather messy operator, but its only property which interests us here is that it is singular at $\zeta^2 = 1$ only,

$$\Delta_\zeta f(\zeta) = f(\zeta q) - f(\zeta q^{-1})\,.$$

q-exact form.

We call $\Delta_\zeta f(\zeta)$ q -exact form if $f(\zeta)=\zeta^\alpha R(\zeta^2)$, and $R(\zeta^2)$ is a rational function with pole at $\zeta^2=1$ only.

Definition of annihilation operators.

$$\bar{\mathbf{c}}(\zeta,\alpha)(X_{[k,l]}):=\frac{1}{2\pi i}\oint\limits_{\Gamma}\psi(\zeta/\xi,\alpha+\mathbb{S})\mathbf{k}(\xi,\alpha)(X_{[k,l]})\frac{d\xi^2}{\xi^2}\,,$$
$$\mathbf{c}(\zeta,\alpha)(X_{[k,l]}):=\frac{1}{4\pi i}\oint\limits_{\Gamma}\psi(\zeta/\xi,\alpha+\mathbb{S})\left\{\mathbf{k}(q\xi,\alpha)+\mathbf{k}(q^{-1}\xi,\alpha)\right\}(X_{[k,l]})\frac{d\xi^2}{\xi^2}\,,$$

$\psi(\zeta)=\zeta^\alpha\frac{\zeta^2+1}{2(\zeta^2-1)}$, and Γ goes around $\zeta^2=1$. Then

$$\mathbf{c}(\zeta,\alpha)(q^{2(\alpha+1)S_{[k,m-1]}}\otimes Y_{[m,l]})=q^{2\alpha S_{[k,m-1]}}\otimes\mathbf{c}(\zeta,\alpha)(Y_{[m,l]})$$
$$\mathbf{c}(\zeta,\alpha)(Y_{[k,m]}\otimes I_{[m+1,l]})=\mathbf{c}(\zeta,\alpha)(Y_{[k,m]})\otimes I_{[m+1,l]}\,,$$

and the same for $\bar{\mathbf{c}}(\zeta,\alpha)$.

Non-violent thermodynamic limit.

Consider the infinite chain, and introduce the space

$$\mathcal{W}^{(\alpha)}=\bigoplus_{s=-\infty}^{\infty}\mathcal{W}_{\alpha-s,s}\,,$$

where $\mathcal{W}_{\alpha-s,s}$ is the space of operators $q^{2(\alpha-s)S(0)}\mathcal{O}^s$ with $\mathcal{O}^s$ being local of spin s.
The inductive limit $k\to-\infty$, $l\to\infty$ of $\mathbf{c}(\zeta,\alpha)$ is well-defined as operator acting from $\mathcal{W}_{\alpha+1,-1}$ to $\mathcal{W}_{\alpha,0}$. Similarly, we define blocks $\mathbf{c}(\zeta,\alpha-s)$ acting from $\mathcal{W}_{\alpha-s+1,s-1}$ to $\mathcal{W}_{\alpha-s,s}$. Combining these blocks we define

$$\mathbf{c}(\zeta):\ \ \mathcal{W}^{(\alpha)}\ \to\ \mathcal{W}^{(\alpha)}\,.$$

The important property of this annihilation operator is that it kills the primary field:

$$\mathbf{c}(\zeta)(q^{2\alpha S(0)})=0\,.$$

The second annihilation operator is lowers spin and raises α. It is defined as follows:

$$\mathbf{b}(\zeta,\alpha)=\phi(\mathbf{c})(\zeta,\alpha)\,,$$
$$\phi(\mathbf{x})(\zeta,\alpha)=q^{-1}N(\alpha-\mathbb{S}-1)\circ\mathbb{J}\circ\mathbf{x}(\zeta,-\alpha)\circ\mathbb{J}\,,\quad N(x)=q^{-x}-q^{x}\,.$$

Creation operators.

The construction of creation operators is a result of experimental work. Its mathematical meaning is a mystery. Consider the reduction relation:

$$\mathbf{k}(\zeta,\alpha)(Y_{[k,m]}\otimes I_{[m+1,l]})=\mathbf{k}(\zeta,\alpha)(Y_{[k,m]})\otimes I_{[m+1,l]}+\Delta_\zeta\mathbf{v}(\zeta,\alpha)(Y_{[k,m]}\otimes I_{[m+1,l]})\,.$$

It gives a temptation to take the "primitive function" $\Delta_\zeta^{-1}\mathbf{k}(\zeta,\alpha)$. The problem here is what to do with singularities? They will lead to transcendental functions. We proceed as follows.

Define

$$\mathbf{f}(\zeta,\alpha)(X_{[k,l]}) = \Delta_\zeta^{-1}\left(\left\{\mathbf{k}(\zeta,\alpha) - \bar{\mathbf{c}}(\zeta,\alpha) - \mathbf{c}(q\zeta,\alpha) - \mathbf{c}(q^{-1}\zeta,\alpha)\right\}(X_{[k,l]})\right).$$

It is clear that $\mathbf{f}(\zeta,\alpha) = \zeta^\alpha R(\zeta^2)$, and $R(\zeta^2)$ is rational with pole at $\zeta^2 = 1$ only. Now we define

$$\mathbf{b}^*(\zeta,\alpha)(X_{[k,l]}) := \left(\mathbf{f}(q\zeta,\alpha) + \mathbf{f}(q^{-1}\zeta,\alpha) - \mathbf{t}^*(\zeta,\alpha)\mathbf{f}(\zeta,\alpha)\right)(X_{[k,l]}).$$

where

$$\mathbf{t}^*(\zeta,\alpha)(X_{[k,l]}) = \mathrm{Tr}_a\left\{\mathbb{T}_{a,[k,l]}(\zeta,\alpha)(X_{[k,l]})\right\}.$$

Reduction relations.

1. The left reduction is the same as before:

$$\mathbf{b}^*(\zeta,\alpha)(q^{2(\alpha+1)S_{[k,m-1]}} \otimes Y_{[m,l]}) = q^{2\alpha S_{[k,m-1]}} \otimes \mathbf{b}^*(\zeta,\alpha)(Y_{[m,l]}),$$

2. The right reduction relation is more complicated:

$$\mathbf{b}^*(\zeta,\alpha)(X_{[k,m]}\otimes I_{[m+1,l]})=\mathrm{Tr}_c\left\{\mathbb{T}_{c,[m+1,l]}(\zeta)\left(\mathbf{g}_c(\zeta,\alpha)(X_{[k,m]})\right)\right\}\ ,$$

where $\mathbf{g}_c(\zeta,\alpha)(X_{[k,m]})$ is regular at $\zeta^2=1$. Recall that

$$\mathbb{T}_{c,[m+1,l]}(\zeta)=\mathbb{L}_{c,l}(\zeta)\cdots\mathbb{L}_{c,m+1}(\zeta)\,.$$

Let us think about $\mathbb{L}_{c,j}(\zeta)$. We know that $\mathbb{L}_{c,j}(0)=\mathbb{P}_{c,j}$. Introduce

$$\widehat{\mathbb{L}}_{c,j}(\zeta)=\mathbb{L}_{c,j}(\zeta)\mathbb{P}_{c,j}\ ,$$

and move the permutation to the right:

$$\begin{aligned}\mathbf{b}^*(\zeta,\alpha)(X_{[k,m]}\otimes I_{[m+1,l]})=\mathrm{Tr}_c\Big\{&\widehat{\mathbb{L}}_{c,l}(\zeta)\widehat{\mathbb{L}}_{l,l-1}(\zeta)\cdots\widehat{\mathbb{L}}_{m+2,m+1}(\zeta)\\ &\times\left(\mathbf{g}_{m+1}(\zeta,\alpha)(X_{[k,m]})\right)\Big\}\ .\end{aligned}$$

Clearly,

$$\widehat{\mathbb{L}}_{i,j}(\zeta) = I + (\zeta^2 - 1)\mathbf{l}_{i,j}(\zeta)\,,$$

with $\mathbf{l}_{i,j}(\zeta)$ being regular at $\zeta^2 = 1$, and

$$\mathbf{l}_{i,j}(\zeta)(I_i \otimes I_j) = 0\,.$$

This leads to the important conclusion:

$$\mathbf{b}^*(\zeta,\alpha)(X_{[k,m]} \otimes I_{[m+1,l]})$$
$$= 2\sum_{n=m+1}^{l}(\zeta^2-1)^{n-m-1}\mathbf{l}_{n,n-1}(\zeta)\mathbf{l}_{n-1,n-2}(\zeta)\cdots\mathbf{l}_{m+2,m+1}(\zeta)\left(\mathbf{g}_{m+1}(\zeta,\alpha)(X_{[k,m]})\right)$$
$$+ (\zeta^2-1)^{l-m}\mathrm{Tr}_c\left\{\mathbf{l}_{c,l}(\zeta)\mathbf{l}_{l,l-1}(\zeta)\cdots\mathbf{l}_{m+2,m+1}(\zeta)\left(\mathbf{g}_{m+1}(\zeta,\alpha)(X_{[k,m]})\right)\right\}\ .$$

So, in the limit $l \to \infty$ the Taylor series are well-defined. Thus we obtain the operator

$$\mathbf{b}^*(\zeta):\ \mathcal{W}^{(\alpha)}\ \to\ \mathcal{W}^{(\alpha)}\,.$$

Similarly,

$$\mathbf{c}^*(\zeta,\alpha)(X_{[k,l]}) := -\phi(\mathbf{b}^*)(\zeta,\alpha)(X_{[k,l]})\,,$$

produces the operator

$$\mathbf{c}^*(\zeta):\ \mathcal{W}^{(\alpha)}\ \rightarrow\ \mathcal{W}^{(\alpha)}\,,$$

with block structure opposite to that of $\mathbf{b}^*(\zeta)$. For the same reasons as before the following operator has nice limit as Taylor series:

$$\mathbf{t}^*(\zeta,\alpha)(X_{[k,l]}) = \mathrm{Tr}_a\big\{\mathbb{T}_{a,[k,l]}(\zeta,\alpha)(X_{[k,l]})\big\}\,,$$

giving rise to block diagonal operator

$$\mathbf{t}^*(\zeta):\ \mathcal{W}^{(\alpha)}\ \rightarrow\ \mathcal{W}^{(\alpha)}\,.$$

Commutation relations.

Strangely enough the proof of the following simple relations is very hard.

The operator $\mathbf{t}^*(\zeta)$ is in the center

$$[\mathbf{t}^*(\zeta_1), \mathbf{t}^*(\zeta_2)] = [\mathbf{t}^*(\zeta_1), \mathbf{c}^*(\zeta_2)] = [\mathbf{t}^*(\zeta_1), \mathbf{b}^*(\zeta_2)] = 0,$$
$$[\mathbf{t}^*(\zeta_1), \mathbf{c}(\zeta_2)] = [\mathbf{t}^*(\zeta_1), \mathbf{b}(\zeta_2)] = 0\,.$$

The rest of the operators $\mathbf{b}$, $\mathbf{c}$, $\mathbf{b}^*$, $\mathbf{c}^*$ are fermionic. The only non-vanishing anti-commutators are

$$[\mathbf{b}(\zeta_1), \mathbf{b}^*(\zeta_2)]_+ = -\psi(\zeta_2/\zeta_1, \alpha)\,, \quad [\mathbf{c}(\zeta_1), \mathbf{c}^*(\zeta_2)]_+ = \psi(\zeta_1/\zeta_2, \alpha)\,,$$

Completeness.

The space $\mathcal{W}^{(\alpha)}$ is created from the primary field $q^{2\alpha S(0)}$ by action of Taylor coefficients at $\zeta^2 = 1$ of $\mathbf{t}^*(\zeta)$, $\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$.

Application.

Consider $\mathfrak{H}_{\mathrm{S}} = \bigotimes\limits_{j=-\infty}^{\infty} \mathbb{C}^2$, $\quad \mathfrak{H}_{\mathbf{M}} = \bigotimes\limits_{\mathbf{j=1}}^{\mathbf{n}} \mathbb{C}^{2s_{\mathbf{j}}+1}$, and define

$$Z^{\kappa}\Bigl\{q^{2\alpha S(0)}\mathcal{O}\Bigr\} = \frac{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\Bigl(T_{\mathrm{S},\mathbf{M}}\,q^{2\kappa S+2\alpha S(0)}\mathcal{O}\Bigr)}{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\Bigl(T_{\mathrm{S},\mathbf{M}}\,q^{2\kappa S+2\alpha S(0)}\Bigr)}\,.$$

We prove that

$$Z^{\kappa}\bigl\{\mathbf{t}^*(\zeta)(X)\bigr\} = 2\rho(\zeta)Z^{\kappa}\{X\}\,,$$

$$Z^{\kappa}\bigl\{\mathbf{b}^*(\zeta)(X)\bigr\} = \frac{1}{2\pi i}\oint\limits_{\Gamma}\omega(\zeta,\xi)Z^{\kappa}\bigl\{\mathbf{c}(\xi)(X)\bigr\}\frac{d\xi^2}{\xi^2}\,,$$

$$Z^{\kappa}\bigl\{\mathbf{c}^*(\zeta)(X)\bigr\} = -\frac{1}{2\pi i}\oint\limits_{\Gamma}\omega(\xi,\zeta)Z^{\kappa}\bigl\{\mathbf{b}(\xi)(X)\bigr\}\frac{d\xi^2}{\xi^2}\,,$$

where Γ goes around $\xi^2 = 1$.