D. Friedan — Introduction to 2d Nonlinear σ-model and Renorm. group flow

Y. Rubinstein — Lower order terms in Sobolev Embedding

I Sobolev Embedding Thm.

Ω - domain in $\mathbb{R}^n$ or a compact manifold

$$W^{1,p}(\Omega) \hookrightarrow \begin{cases} L^{\frac{np}{n-p}}(\Omega), & \text{if } p<n \\ C^{0,\alpha}(\Omega), \text{ for } \alpha \leq 1-\frac{n}{p}, & \text{if } p>n \end{cases}$$

$$C^\infty \cap L^q(\Omega) = \{u \in C^\infty(\Omega) : \int |u|^q < \infty\}$$

$u \in W^{1,p}$ if: $|\nabla u| \in L^p$ & $u \in L^p$

Case 3: $p = n$

Moser-Trudinger Thm: $\Omega = M = S^2$ $\quad n=p=2$

that says $\varphi \in W^{1,2}(S^2) \Rightarrow e^\varphi \in L^1(S^2)$

$\to \|u\|_{L^{\frac{np}{n-p}}} \leq C\|u\|_{W^{1,p}}, \ p<n$

$$\int_{S^2} e^u w \leq C_1 e^{C_2 \|u\|^2_{W^{1,2}}}$$

w is a vol form on S^2 (true for any domain in $\mathbb{R}^2$ and compact m.)

II <u>Question</u> Does the SETh admit lower order corrections? Consider S^2.

Know that $\|u\|_{W^{1,2}} \leq C \rightsquigarrow \|e^u\|_{L^1} < \tilde{C}$

Q: $\|u\|_{W^{1,2}} - C_1\|u\|_{W^{1,2}} \leq C \Rightarrow \|e^u\|_{L^1} \leq \tilde{C}$

Why interesting?

$(S^2, \overset{\text{c.s.}}{J})$ w Kähler form = vol. form

$$(w + i\partial\bar{\partial}\varphi)^m = e^{f_w - \varphi} w^m \quad (\text{KE eqn.})$$

$m = \frac{n}{2} = 1$

$$\text{Ric}(w + i\partial\bar{\partial}\varphi) = w + i\partial\bar{\partial}\varphi$$

$$F(\omega, \omega_\varphi) = \frac{1}{V} \int_{S^2} \frac{1}{2} i \, \partial\varphi \wedge \bar\partial\varphi - \frac{1}{V} \int_{S^2} \varphi \omega -$$

$$- \log \frac{1}{V} \int_{S^2} e^{f_\omega - \varphi} \omega \qquad V = \int_{S^2} \omega$$

↗ Variation produces previous eqn.

Assume ω is round metric $\Rightarrow$ $f_\omega = 0$

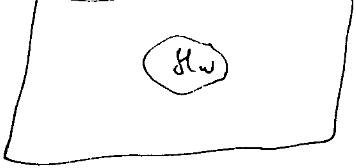
expect that $F(\omega, \omega_\varphi) \geq 0$ (proved already)

If we knew $F \geq 0$

$$\frac{1}{V} \int e^{-\varphi} \omega \leq e^{\frac{1}{V} \int \frac{1}{2} i \partial\varphi \wedge \bar\partial\varphi - \frac{1}{V} \int \varphi\omega}$$

φ is not arb. f-n — must be Kähler potential, i.e.:

$C^\infty(S^2)$



$$\mathcal{H}_\omega = \{ \varphi \in C^\infty(S^2) : \omega + i \partial\bar\partial\varphi > 0 \}$$

III Consider another functional E_n (aside $E_0, E_1, \dots E_n$ Chen-Tian funct.)

k-Energy

$$dE_n\Big|_\alpha = \left(\Delta_\alpha \frac{(\mathrm{Ric}\,\alpha)^n}{\alpha^n} \right) \alpha^n$$

$\mathcal{H}_\omega \subseteq C^\infty(S^2)$ $\quad T_\varphi \mathcal{H}_\omega = C^\infty(S^2)$ $\quad T^*_\varphi \mathcal{H}_\omega = \Omega^n(S^2)$

top-forms

for α assign $dE_n|_\alpha$

$$\mathrm{Ric}^{-1}: C^\infty(M) \to \mathcal{H}_\omega$$

$$\psi \to \varphi \quad \text{s.t. } \mathrm{Ric}\,\omega_\varphi = \omega_\psi$$

Prop

$(\mathrm{Ric}^{-1})^* E_n = F$ $\quad$ (on S^2 take E_1)

If $E_1 \geq 0$ on $\mathcal{H}_\omega$, then Prop $\Rightarrow$ $F \geq 0$ on $C^\infty(S^2)$

Thm (Song-Weinkove)

Corollary of $E_0 \geq 0$ on $\mathcal{H}_\omega$ (Bando-Mabuchi)

$$J(\omega, \omega_\psi) = \frac{1}{V} \int \frac{1}{2} i \, \partial\psi \wedge \bar\partial\psi$$

$$\frac{1}{V} \int e^{-\varphi} \omega \le e^{\frac{1}{V} \int \frac{1}{2} i \, \partial\varphi \wedge \bar\partial\varphi - \frac{1}{V} \int \varphi\omega - J(Ric^{-1}\omega_\varphi, \omega_\varphi)}$$

Onofri – connection to Liouville theory