S. Zelditch

$$\mathcal{H} = \{ \varphi \in C^\infty(M) : \omega_0 + i\partial\bar\partial\varphi > 0 \}$$

positive (1,1) form

Space of Kähler metrics in a fixed Kähler class

Assume $\frac{1}{2\pi}[\omega_0] \in H^{1,1}(M, \mathbb{Z}) \Rightarrow$

$$\exists\ L \to M, \text{ s.t. } c_1(L) = [\omega_0]$$

φ gives rise to Hermitean metric

$$\varphi \rightsquigarrow e^{-\varphi} h_0 \quad \text{Herm. metric on } L$$

$$i\partial\bar\partial \log h_0 = \omega_0$$

Geometry of $\mathcal{H}$:

$\exists$ Riemannian structure $\quad g_\varphi(\delta\varphi_1, \delta\varphi_2) =$

$$T_\varphi \mathcal{H} \simeq C^\infty(M)$$

$$= \int_M \delta\varphi_1(z)\,\delta\varphi_2(z)\, d\,\mathrm{Vol}_\varphi$$

$$d\mathrm{Vol}_\varphi = \frac{\omega_\varphi^n}{n!}$$

Kähler Quantization

applies to any symplectic manifold (compact)

$$(M, \omega_0)$$

Hilbert space:

$H^0(M, L^k)$ - space of holomorphic sections of L^k

$$\dim H^0(L^k) = d_k + 1 \sim a_0 k^m + a_1 k^{m-1}$$

Ex $M = \mathbb{CP}^m$
$L = \mathcal{O}(1)$
$L^k = \mathcal{O}(k)$

$H^0(M, L^k) \cong$ homogeneous poly's on $\mathbb{C}^{m+1}$ of degree k

Herm. met. on $L^1 \rightsquigarrow h^{\otimes k} = h^k$

Given $h \to$ induces $Hilb_k(h)$ (inner products)

$$\langle s_1, s_2 \rangle = \int_M (s_1(z), s_2(z))_{h^k} \, d\mu_h$$

Locally, $\exists$ frame : $e^k : U \to L^k$
$U \subset M$

section $s = f e^k$; $f \in \mathcal{O}(U)$

then $\langle s_1, s_2 \rangle = \int_M f_1(z) \overline{f_2(z)} e^{-k\varphi} d\mu_\varphi$

Bergman (= Szegö) Kernel

$\Pi_{h^k} : L^2(M, L^k) \to H^0(M, L^k)$

orthogonal w.r.t. $Hilb_k(h)$

Quantization

functions $C^\infty(M, \mathbb{R}) \rightsquigarrow$ hermitean operators on $H^0(M, L^k)$

Symplectic map $\rightsquigarrow$ unitary operator
of (M, ω)

$\hbar = \frac{1}{k}$

$k \to \infty$ "classical limit"

$a \in C^\infty(M, \mathbb{R}) \rightsquigarrow \underbrace{\Pi_{h^k} a \Pi_{h^k}}_{\text{Toeplitz operator}}$

$\Pi_{h^k}^2 = \Pi_{h^k} = \Pi_{h^k}^*$

Let I_k = Herm inner product on $\Gamma(M, L^k)$

$\Pi_{h^k} a \Pi_{h^k} s(z) = \Pi_{h^k} a s(z)$

hol. section

Proposition $\lim_{k \to \infty} \frac{1}{d_k} \operatorname{Tr} \Pi_{h^k} a \Pi_{h^k} =$

$$= \frac{1}{\operatorname{Vol} M} \int_M a \, d\mu_h \qquad \left(\text{for } Hilb_k(h) \text{ sequence of inner pr.} \right)$$

Proof Kernel : $As(z) = \int A(z,w) \overline{s(w)} \, d\mu_h$

Kernel of Π: $\Pi_{h^k}(z,w) = \sum_{\alpha=0}^{d_k} S_\alpha(z) \otimes S_\alpha(w)^*$,

$\{s_\alpha\}$ - ONB for $Hilb_k$

Kernel $\Pi_{h^k} a = \cancel{a(z)} \int \Pi_{h^k}(z,w) \, a(w) \, \cancel{s(w)}$

$$\frac{\operatorname{Tr} \Pi_{h^k} a}{\operatorname{Tr} \Pi_{h^k}} = \int_M \Pi_{h^k}(z,z) \, a(w) \, d\mu_h \sim \frac{\int a(w) \, d\mu_h}{\int d\mu_h}$$

TYZ expansion $\Pi_{h^k}(z,z) \sim k^m \left(1 + \frac{S(z)}{k} + \frac{S_2(z)}{k^2} + \dots \right)$

as $k \to \infty$, can be differentiated any # of times

$\operatorname{Tr} \Pi_{h^k} = d_k + 1$

S = scalar curvature of ω_h

S_2 = curvature inv $s^2, \Delta S, \ldots$

<u>Prop</u> $\dfrac{\operatorname{Tr}(\Pi_{h^k} a \Pi_{h^k})^2}{\operatorname{Tr}\Pi_{h^k}} = \Pi_{h^k} a \Pi_{h^k}^2 a \Pi_{h^k} =$

$$= \iint \Pi_{h^k}(z,w)\, a(w)\, \Pi_{h^k}(w;z)\, a(z)\, d\mu_h\, d\mu_h$$

$$\sim \int a^2 d\mu_h$$

(Szegö limit theorem)

Boutet de Mouvel - Sjostrand - Fefferman

$$\Pi_{h^k}(z,w) = B_{h^k}(z,w)\, e^k(z) \otimes e^k(w)^*$$

$$B_{h^k}(z,w) \sim e^{k\varphi(z,w)} A_k(z,w)$$ ("WKB" formula) = "paramatrix"

$$\varphi(z,\bar z) \sim \varphi(z,\bar w)$$ (replace)

e.g. $\varphi(z) = \log(1+|z|^2)$

$\varphi(z,w) = \log(1+z\bar w)$

symbol or poly-homogeneous amplitude

$A_k(z,w) \sim \sum k^{m-j} \partial a_j(z,w)$

Bargmann - Fock $\quad \varphi_{BF}(z) = |z|^2 \quad \varphi(z,w) = z\bar{w}$

Calabi's diastasis function :

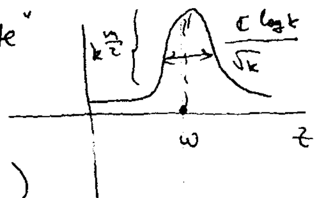
$$\varphi(z,w) + \varphi(w,z) - \varphi(z,z) - \varphi(w,w)$$

natural distance between points

$$\Pi_{h^k}(z,w) \sim e^{k(\varphi(z,w) - \frac{1}{2}\varphi(z) - \frac{1}{2}\varphi(w))}$$

$$P_{h^k}(z,w) = \frac{\Pi_{h^k}(z,w)}{\sqrt{\Pi_{h^k}(z,z)\,\Pi_{h^k}(w,w)}} = \Phi^w_{h^k}(z) \sim e^{-k|z-w|^2}$$

"coherent state"



embed $W \to \Phi^w_{h^k} \in H^0(\mathbb{CP}^n, L^k)$

$$\Pi_{h^k}(z,w) = e^{k(\overbrace{\varphi(z,w)}^{\text{normal coordinates}} - \frac{1}{2}\varphi(z) - \frac{1}{2}\varphi(w))} A$$

$$\varphi(z) \sim |z|^2 + \sum R_{i\bar{j}k\bar{l}}\, z_i \bar{z}_j z_k \bar{z}_l$$

Symplectic maps: quantize Hamiltonian flows

H generates Hamiltonian flow $\qquad \hat{H} = \Pi_{hk}\, a\, \Pi_{hk}$

$$X^{\omega}_{H} = \tilde{\omega}^{-1}(dH) \quad , \quad \tilde{\omega} : TM \to T^{*}M$$
$$X \to i_X \tilde{\omega}$$

$$\exp t X^{\omega}_{H} \quad \approx> \quad e^{itk\hat{H}} = U_k(t)$$

$$e^{-itk\hat{H}}\, \Pi_{hk}\, a\, \Pi_{hk}\, e^{itk\hat{H}} \sim \Pi_{hk} \exp t X_H a\, \Pi_{hk}$$

(analogously $\{\ \} \to i[\ ,\ ]$)

Back to $\mathcal{H}$. It's symmetric space $G_{\mathbb{C}}/G$

$G = SDiff_{\omega_0}(M)$

Hamilton(M)

$\exp it X^{\omega}_{H}$

Bergman metrics of level k $\quad \mathcal{B}_k \subset \mathcal{H}$

$\mathcal{I}_k$ = Inner products on $H^0(M, L^k)$

$FS_k : \mathcal{I}_k \to \mathcal{H}$

$G \sim$ ONB $\to i_S = [f_0 : \cdots : f_{d_k}]$
$\{S_\alpha\}$

$\mathcal{B}_k$ Bergman metrics: $= \{ i_S^* \omega_{FS}$
$S \in \text{Bases}_{\#} H^0(L^k) \}$

$$FS_k(G) = \frac{1}{k} \log |\vec{s}(z)|$$

$$\begin{array}{ccc} B_k & \hookrightarrow & \mathcal{H} \\ & & \downarrow FS_k \circ Hilb_k \\ & & B_k \end{array}$$

$$B_k = GL(d_k+1, \mathbb{C}) / U(d_k+1)$$

Program:

Would like to understand geometry of $\mathcal{H}$ via approx. by B_k

What would we like to know:

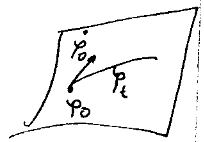
0) : points of $\mathcal{H}$ are well approx. by points of B_k

$$h \in \mathcal{H} \rightsquigarrow \underset{\substack{\| \\ h_k}}{FS_k \circ Hilb_k(h)} \in B_k$$

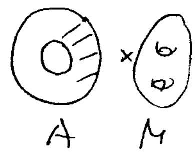
$$\frac{h_k}{h} = 1 + O\left(\frac{1}{k}\right)$$

02: Approximation of geodesics (Phong-Sturm)

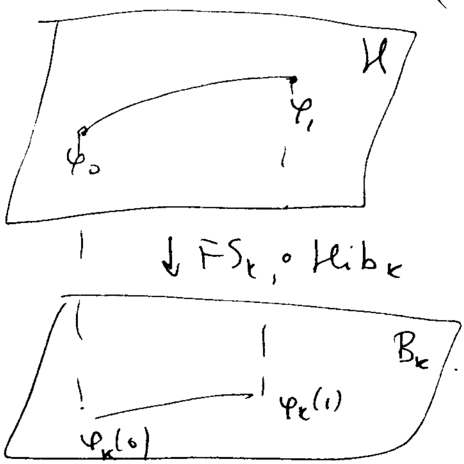
paths φ_t in $\mathcal{H}$



$$\ddot{\varphi}_t = \frac{1}{2} |\nabla_t \dot{\varphi}|^2_{\omega_t} \iff (\partial\bar\partial\Phi)^{m+1} = 0 \text{ on } A \times M$$



Dirichlet problem (endpoint) Monge-Ampère equ.

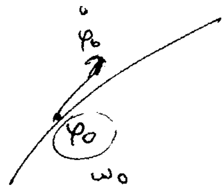


Thm (Ph-St)

B_k geodesic $\xrightarrow{C^0 \text{ weak}}$ MA geod. as $k \to \infty$

$$\frac{1}{k} \log \sum_{\alpha=0}^{d_k} e^{t\lambda_\alpha} |S_\alpha(z)|^2$$

Donaldson's formal solution for IV Prob.



Let $\exp tX^{\omega_0}_{\dot\varphi_0}$ = Ham. flow

then $\left(\exp itX^{\omega_0}_{\dot\varphi_0}\right)^* \omega_0 = \omega_0 + i\partial\bar\partial\varphi_t$

geod.

(can't analitically continue!)

~~Alg~~ <u>Strategy</u> to construct φ_t

- quantize Ham. flow: $e^{itk\,\Pi_{h^k}\dot\varphi\,\Pi_{h^k}} = U_k(t)$ on $H^0(L^k)$
- analitically cont. $U_k(t) \sim e^{-tk\,\Pi_{h^k}\dot\varphi\,\Pi_{h^k}} = U_k(it)$
- $\lim_{k\to\infty} \frac{1}{k} \log U_k(it,z,z) = \varphi(t,z)$ - geodesic

(Conjecture Rubinstein-SZ)

<u>Theorem</u>: this works for toric Kähler manifolds

Toric Kähler manifold

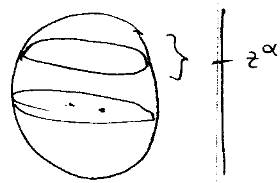
$(\mathbb{C}^*)^m$ action w/ open orbit

$$H^0(M, L^k) \cong \{ z^\alpha, \alpha \in kP \}$$

polytope

$$\frac{1}{k} \log U_k(it, z, z) = \frac{1}{k} \log \sum_{\alpha \in kP} e^{k\mu(\alpha)} |z^\alpha|^2$$

$$e^{itk\hat{H}} z^\alpha = e^{itk\mu(\alpha)} z^\alpha$$

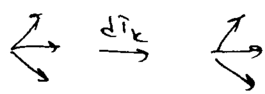


$$\hat{H} = \Pi_{h^k} \dot{\varphi} \, \Pi_{h^k}$$

Curvature tensor for B_k, $X, Y, Z \in TB_k$

$-\frac{1}{4}[[X, Y], Z]$ converges to C.t. on $\mathcal{H}$

$T_k = FS \circ Hilb_k \quad \mathcal{H} \to B_k$



$$-\frac{1}{4}\left[[dT_k(X), dT_k(Y)], dT_k(Z)\right] \to -\frac{1}{4}\{\{X, Y\}, Z\}$$