A Bird Eye's view: recent update to Extremal metrics

Xiuxiong Chen

Department of Mathematics
University of Wisconsin at Madison

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Based on work of myself and students.

Basic setup in Kähler Geometry

$(M, [\omega])$ is a polarized Kähler manifold where

$$\omega = \frac{\sqrt{-1}}{2} \sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}}\, d\, w_\alpha \wedge d\, \bar{w}_\beta > 0 \qquad \text{on } M.$$

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In some local coordinate $U \subset M$, there is a local potential function ρ such that

$$g_{\alpha\bar{\beta}} = \frac{\partial^2 \rho}{\partial w_\alpha \partial \bar{w}_\beta}, \qquad \forall\ \alpha, \beta = 1, 2, \cdots n.$$

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A Kähler class

$$[\omega] = \{\omega_\varphi \mid \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \text{ on } M\}$$

where φ is a real valued function.

Ricci form:

$$\begin{aligned} Ric(\omega) &= -\sqrt{-1}\partial\bar{\partial}\log\omega^n \\ &= -\sqrt{-1}\partial\bar{\partial}\ \log\ \det\left(g_{\alpha\bar{\beta}}\right). \end{aligned}$$

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Scalar curvature:

$$\begin{aligned} R &= -g^{\alpha\bar{\beta}}\frac{\partial^2}{\partial w_\alpha \bar{\partial} w_\beta}\log\det\left(g_{\alpha\bar{\beta}}\right) \\ &= -\triangle_g \log\det\left(g_{\alpha\bar{\beta}}\right). \end{aligned}$$

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The first Chern class is positive definite (resp: negative definite) if

$$[Ric(\omega)] > (\text{resp.} <)\ 0\ \text{ on } M.$$

Calabi's program

$$Ca(\omega_\varphi) = \int_M \left(R(\omega_\varphi) - \underline{R}\right)^2 \ \omega_\varphi^n.$$

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- Critical points
  - Extremal Kähler metric: $\mathcal{X}_c = g^{\alpha\bar{\beta}} \frac{\partial R}{\partial \bar{w}_\beta} \frac{\partial}{\partial w_\alpha}$ is holomorphic.
  - Constant Scalar curvature metric: $R(\omega_\varphi) = const.$
  - Kähler-Einstein metric:$Ric(\omega) = \lambda\omega$; where $\lambda = 1, 0, -1$.

The space of Kähler potentials

$$\mathcal{H} = \{\varphi \mid \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \text{ on } M\}$$

$$T_\varphi\mathcal{H} = C^\infty(M).$$

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The geodesic equation is:

$$\varphi''(t) - g_\varphi^{\alpha\bar{\beta}} \varphi'_\alpha(t)\varphi'_{\bar{\beta}}(t) = 0 \tag{1}$$

where $g_{\varphi_{\alpha\bar{\beta}}} = g_{\alpha\bar{\beta}} + \frac{\partial^2\varphi}{\partial w_\alpha \partial \bar{w}_\beta}$.

According to S. Semmes, this can be written as

$$\det\left(g_{i\bar{j}} + \varphi_{i\bar{j}}\right)_{(n+1)\times(n+1)} = 0$$

in $[0,1] \times S^1 \times M$.
It is called a **geodesic ray** if (1) holds in $[0,\infty) \times S^1 \times M$.

Curvature of $\mathcal{H}$

Let $\varphi(t)$ be a path in $\mathcal{H}$, one can define a connection in $\mathcal{H}$ as:

$$\nabla_{\frac{\partial}{\partial t}} v = \frac{\partial v}{\partial t} - \frac{1}{2}(\nabla \frac{\partial \varphi}{\partial t} \cdot \nabla v)_{\varphi}$$

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T. Mabuchi, S. Semmes, S. K. Donaldson showed this Levi-civita and proved formally that for $\xi, \zeta, \psi \in T_v\mathcal{H}$, then

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In particular, the sectional curvature is non-negative

$$R(\xi, \zeta, \zeta, \xi) = -\frac{1}{4}|\{\xi, \zeta\}_v|_v^2 \leq 0.$$

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Denote by N_k the dimension of $H^0(X, L^k)$. Let B_k be the space of positive definite Hermitian matrixes in $H^0(X, L^k)$.

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Then B_k could be viewed as the non-compact symmetric space $GL(N_k;\mathbb{C})/U(N_k)$, with the standard metric on matrices. The spaces B_k are related to $\mathfrak{H}$ through two naturally defined maps:

$$Hilb_k : \mathcal{H} \to B_k, \qquad \text{and} \quad FS_k : B_k \to \mathcal{H}$$

Theorem

(Chen-Sun) Given any $\phi_0, \phi_1 \in H$, the distance in $\mathfrak{B}_k$ of $Hilb_k(\phi_0)$ and $Hilb_k(\phi_1)$ divided by $k^{\frac{n+2}{2}}$ converges to the distance of ϕ_0 and ϕ_1 in $\mathfrak{H}$ as $k \to \infty$.

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Using Kodaira embedding to approximate geodesic: Phong-Sturm, Song-Zelditch, Rubinstein-Zelditch.

The K energy functional

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where $(\varphi, \psi) \in T\mathcal{H}$. Clearly, the Euler Lagrange equation of this functional is

$$R(\omega_\varphi) - \underline{R} = constant.$$

Donaldson's Conjecture

- $\forall\ \varphi_1, \varphi_2 \in \mathcal{H}, \exists! \mathcal{C}^\infty$ geodesic $\varphi(t) \in \mathcal{H}$ which connects φ_1 and φ_2.
- $\mathcal{H}$ is a metric space, i.e., $d(\varphi_1, \varphi_2) > 0$ if $\varphi_1 \neq \varphi_2$.
- The K energy is a convex function along geodesic. ($\Rrightarrow$ Extremal Kähler metric is unique.)

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X. X. Chen (1998) proved that $\mathcal{H}$ is geodesically convex by $C^{1,1}$ geodesic.
X. Chen + Calabi (1999) proved that $\mathcal{H}$ is non-positively curved.

Theorem

(Chen) For any two Kähler metric $\varphi_0, \varphi_1 \in \mathcal{H}$*, and the unique* $C^{1,1}$ *geodesic* $\varphi(t,x)$ *connecting these two metrics such that* $\varphi(0,x) = \varphi_0$ *and* $\varphi(1,x) = \varphi_1$*, then*

$$\left(d\mathbf{E} \mid_{\varphi_0}, \frac{\partial \varphi}{\partial t} \mid_{t=0} \right) \leq \left(d\mathbf{E} \mid_{\varphi_1}, \frac{\partial \varphi}{\partial t} \mid_{t=1} \right).$$

Theorem

(Chen) For any $\omega_\varphi \in [\omega]$, *we have*

$$Ca(\omega_\varphi) \geq \mathcal{F}_{[\omega]}(\mathcal{X}_c) = ||\mathcal{F}_{[\omega]}||^2.$$

Remark: S. K. Donaldson proved similar inequality in algebraic case.

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Theorem

(Chen-Tian) The K energy functional bounded from below if there exists a cscK metric in its Kähler class.

In the space of Kahler potentials, we have

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the infimum of Calabi energy in a Kähler class is 0 if and only if the K energy in the same class is uniformly bounded from below.

Such a statement is false in finite dimension Euclidean space with respect to convex functions. However, it will be interesting to see what happen in infinite dimension with nice structure.

Yau-Tian-Donaldson conjecture

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An algebraic polarization is called **K stable** if the generalized futaki invariant in the central fiber in any non-trivial test configuration is negative.

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In algebraic manifold $(M, [\omega])$, the existence of cscK metric is equivalent to the K stability of $(M, [\omega])$.
An algebraic polarization is called **K stable** if the generalized futaki invariant in the central fiber in any non-trivial test configuration is negative.
This is first introduced by G. Tian in 1997 in terms of special degeneration and extended for more general setting by Donaldson.

Definition

Let $L \to M$ be an ample line bundle over a compact complex manifold. A test configuration $\mathcal{M}$ consists of:

1. a scheme $\mathcal{M}$ with a C^*-action.
2. a C^*-equivariant line bundle $\mathcal{L} \to \mathcal{M}$.
3. a flat C^*-equivariant map $\pi : \mathcal{M} \to C$, where C^* acts on C by multiplication.

and any fiber $M_t = \pi^{-1}(t)$ for $t \neq 0$ is isomorphic to M.

Theorem

For any destabilized simple test configuration such that the central fibre admits a cscK metric, then the K-energy functional in the Kähler class defined by nearby fibre is bounded from below uniformly.

Geodesic rays

Definition

For every smooth geodesic ray $\rho(t)(t \in [0, \infty))$, we can define an invariant as

$$\yen(\rho) = \lim_{t \to \infty} \int_M \frac{\partial \rho(t)}{\partial t}(\underline{R} - R(\rho(t)))\omega^n_{\rho(t)}. \tag{2}$$

Definition

A smooth, special geodesic ray $\rho(t) : [0, \infty) \to \mathcal{H}$ is called stable (resp; semi-stable) if $\yen(\rho) > 0$ (resp: ≥ 0). It is called a destabilizer for $\mathcal{H}$ if $\yen(\rho) < 0$.

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Definition

A Kähler manifold is called geodesically stable if $\yen > 0$ along any smooth geodesic ray. It is called weakly geodesically stable if there is no destabilizing smooth geodesic ray.

Theorem

If there exists a smooth geodesic ray $\rho(t) : [0, \infty) \to \mathcal{H}$ *which is tamed by an ambient geometry, then for any Kähler potential* $\varphi_0 \in \mathcal{H}$*, there exists a relative* $C^{1,1}$ *geodesic ray* $\varphi(t)$ *initiated from* φ_0 *and parallel to* $\rho(t)$*.*

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Definition

Tamed by a bounded ambient geometry A smooth geodesic ray $(M, \omega_{\rho(t)})$ is called tamed by an ambient metric $h = \omega + \partial\bar{\partial}\bar{\rho}$ if there is a uniform bound of the relative potential $\rho - \bar{\rho}$ and $Riem(h) \geq -C$.

In algebraic polarization, the geodesic stable implies K stable?

Conjecture on existence

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Recently, J. Stoop, cscK implies K stable if $Aut(M)$ is discret.

Existence side of story

The existence by PDE methods

- 1976
  - $C_1 = 0$, S. T. Yau, Calabi-Yau metric.
  - $C_1 < 0$, Existence of Kähler-Einstein metric, S. T. Yau and T. Aubin independently.

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- 2007, X. Chen, C. LeBrun and B. Weber: There is a conformal Einstein metric in every Fano surface.
- 2008, S. K. Donaldson, In Toric Kähler surface, the existence of cscK metric if and only if it is K stable among toric invariant Kähler metrics.

main method of attacks

- continuous method;
- Kähler Ricci flow in canonical class:

$$\frac{\partial g_{i\bar{j}}(t)}{\partial t} = g_{i\bar{j}} - 2R_{i\bar{j}}(g(t)).$$

- The Calabi flow:

$$\frac{\partial \varphi}{\partial t} = R_{\varphi} - \underline{R}.$$

Kähler Ricci flow in Fano surface

Theorem

X. Chen + B. Wang: In any Fano surface with reductive automorphism group, the Kähler Ricci flow initiated from any Kähler metrics will converges to a Kähler Einstein metric exponentially fast.

Calabi flow in Riem. Surface

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$$\frac{\partial u}{\partial t} = \frac{1}{e^{2u}} \triangle_0 \, K(u).$$

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Here

$$K(u) = \frac{-\triangle_0 u + K_0}{e^{2u}}.$$

This is the "Calabi flow."

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These functionals are non-increasing along the Calabi flow.

$$\frac{d}{d\,t} Ca(g(t)) = -\int_M |K, zz|^2 \, d\, g,$$

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Along the Calabi flow, we have

$$\int_M \frac{(-\triangle_0 u + K_0)^2}{e^{2u}} + \int_M e^{2u} \le C.$$

Global Structure of Robinson-Trautman space time $(\mathbb{R}^+ \times R \times M^2)$

There exists a coordinate system in which the metric takes the form

$$
\begin{aligned}
d\,s^2 &= -\Phi d\,u^2 - 2d\,u\,d\,t + t^2 e^{2\lambda} g_{0\,ab} d\,x^a d\,x^b, \\
\Phi &= \frac{1}{2}K + \frac{t}{12m}\triangle_g K - \frac{2m}{t}.
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P. T.Chrusciel, 1991: Assuming uniformization theorem, the Calabi flow exists all time and converges to a constant scalar curvature metric exponentially.

What about higher dimensional?