Global aspects of T-duality

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References

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Motivation

In order to construct M-theory/String Theory axiomatically, we need to know

- **Degrees of freedom:** Perturbative string spectrum (graviton, gauge fields), D-branes (K-theory, gerbes)
- **Symmetries:** Besides usual general coordinate invariance, gauge symmetry, . . . , String Theory possesses additional discrete symmetries (T-duality, S-duality, . . .).

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For a QFT on a manifold M the degrees of freedom are locally (i.e. on a coordinate patch $U_\alpha \subset M$) given in terms of fields $g_{\mu\nu}, A_\mu, B_{\mu\nu}, \ldots$, related in overlaps by coordinate transformations, gauge transformations, etc.

Globally, we can think of these fields as sections of bundles over M, connections on vector bundles, gerbe connections, ...

In String Theory we may also allow for the fields in overlaps to be related by the additional discrete symmetries (T-duality, S-duality, ...). In that case the underlying manifold is no longer geometric. What is the global meaning?

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There are two proposals (based on considering examples of T-duality)

- Noncommutative geometry (open strings): The "nongeometric manifolds" are continuous fields of noncommutative tori over a base manifold M (prize to pay: not locally trivial).
- Generalized geometry (closed strings): The "nongeometric manifolds" are so-called T-folds (prize to pay: doubling of the dimensions) [Hull].

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Closed string on $M \times S^1$

Closed strings on $M \times S^1$ are described by

$$X : \Sigma \to M \times S^1$$

where $\Sigma = \{(\sigma, \tau)\}$ is the closed string worldsheet.
Upon quantization, we find

- Momentum modes: $p = \frac{n}{R}$
- Winding modes: $X(0, \tau) \sim X(1, \tau) + mR$

$$E = \left(\frac{n}{R}\right)^2 + (mR)^2 + \text{osc. modes}$$

We have a duality $R \to 1/R$, such that ST on $M \times S^1$ is equivalent to ST on $M \times \widehat{S}^1$ (or a duality between IIA and IIB ST, for susy ST)

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The Buscher rules

Low energy effective action given by (conformally invariant) σ-model

$$S = \int \left[\sqrt{h} h^{\alpha\beta} g_{MN}(X) \partial_\alpha X^M \partial_\beta X^N + \epsilon^{\alpha\beta} B_{MN}(X) \partial_\alpha X^M \partial_\beta X^N + \sqrt{h} R(h) \Phi(X) \right]$$

Now, suppose we have a $U(N)$ isometry $X^m \to X^m + \epsilon^m$, then this action has a symmetry given by the Buscher rules

$$\widehat{Q}_{MN} = \begin{pmatrix} \widehat{Q}_{\mu\nu} & \widehat{Q}_{\mu n} \\ \widehat{Q}_{m\nu} & \widehat{Q}_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} Q_{\mu\nu} - Q_{\mu m}(Q^{-1})^{mn} Q_{n\nu} & -Q_{\mu m}(Q^{-1})^{m}{}_{n} \\ (Q^{-1})_{m}{}^{n} Q_{n\nu} & (Q^{-1})_{mn} \end{pmatrix}$$

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More explicitly, for a $U(1)$ isometry,

$$
\begin{aligned}
\widehat{g}_{\bullet\bullet} &= \frac{1}{g_{\bullet\bullet}} \\
\widehat{g}_{\bullet\mu} &= \frac{B_{\bullet\mu}}{g_{\bullet\bullet}} \\
\widehat{g}_{\mu\nu} &= g_{\mu\nu} - \frac{1}{g_{\bullet\bullet}} \left(g_{\bullet\mu} g_{\bullet\nu} - B_{\bullet\mu} B_{\bullet\nu} \right) \\
\widehat{B}_{\bullet\mu} &= \frac{g_{\bullet\mu}}{g_{\bullet\bullet}} \\
\widehat{B}_{\mu\nu} &= B_{\mu\nu} - \frac{1}{g_{\bullet\bullet}} \left(g_{\bullet\mu} B_{\bullet\nu} - g_{\bullet\nu} B_{\bullet\mu} \right)
\end{aligned}
$$

Principal S^1-bundles

Suppose we have a pair (E, H), consisting of a principal circle bundle

$$\begin{array}{ccc} S^1 & \longrightarrow & E \\ & & \downarrow \pi \\ & & M \end{array}$$

and a so-called H-flux H, a Čech 3-cocycle.

Topologically, E is classified by an element in $F \in H^2(M, \mathbb{Z})$ while H gives a class in $H^3(E, \mathbb{Z})$

Result

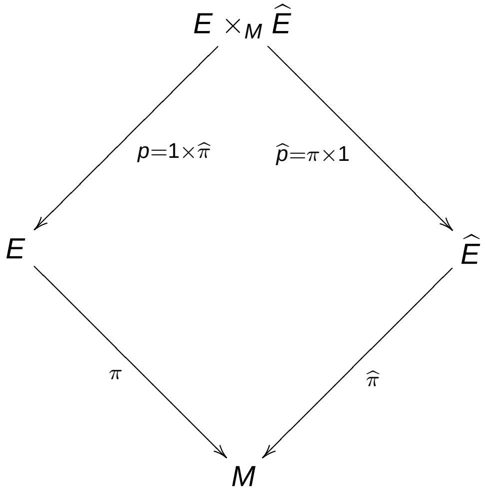
The T-dual of (E, H) is given by the pair $(\widehat{E}, \widehat{H})$, where the principal S^1-bundle

$$\begin{array}{ccc} \widehat{S}^1 & \longrightarrow & \widehat{E} \\ & & \Big\downarrow \widehat{\pi} \\ & & M \end{array}$$

and the dual H-flux $\widehat{H} \in H^3(\widehat{E}, \mathbb{Z})$, satisfy

$$\boxed{\widehat{F} = \pi_* H \,, \qquad F = \widehat{\pi}_* \widehat{H}}$$

where $\pi_* : H^3(E, \mathbb{Z}) \to H^2(M, \mathbb{Z})$, and $\widehat{\pi}_* : H^3(\widehat{E}, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ are the pushforward maps ('integration over the S^1-fiber')


$E \times_M \widehat{E}$
$p = 1 \times \widehat{\pi}$
$\widehat{p} = \pi \times 1$
E
$\widehat{E}$
π
$\widehat{\pi}$
M

The ambiguity in the choice of $\widehat{H}$ is removed by requiring that

$$p^*H - \widehat{p}^*\widehat{H} \equiv 0$$

in $H^3(E \times_M \widehat{E}, \mathbb{Z})$, where $E \times_M \widehat{E}$ is the correspondence space

$$E \times_M \widehat{E} = \{(x, \widehat{x}) \in E \times \widehat{E} \mid \pi(x) = \widehat{\pi}(\widehat{x})\}$$

Locally, the transformation rules on the massless low-energy effective fields (g_{MN}, B_{MN}) are consistent with the Buscher rules.

In particular, since we claim that (type IIA/B) String Theory on E, in the presence of a background H-flux H, is T-dual to (type IIB/A) String Theory on $\widehat{E}$, with background H-flux $\widehat{H}$, the spectrum of D-branes should coincide.

Theorem: This T-duality gives rise to an isomorphism between the twisted K-theories of (E, H) and $(\widehat{E}, \widehat{H})$ (with a shift in degree by 1)

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Examples

- IIA/B ST on S^3 with no H-flux is equivalent to IIB/A ST on $S^2 \times S^1$ with one unit of H-flux
- IIA/B ST on $S^3 \cong SU(2)$ with two units of H-flux is equivalent to IIB/A ST on $\mathbb{R}P^3 \cong SO(3)$ with one unit of H-flux (Cf. Geometric Langlands).

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THANKS