Quantum Cellular Automata from Lattice Field Theories

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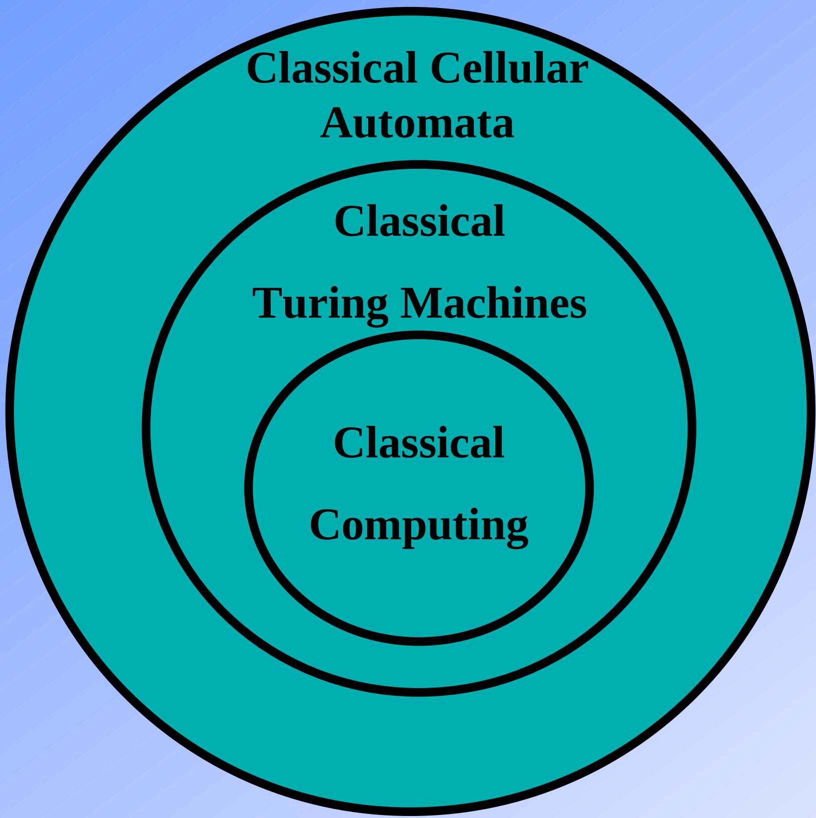
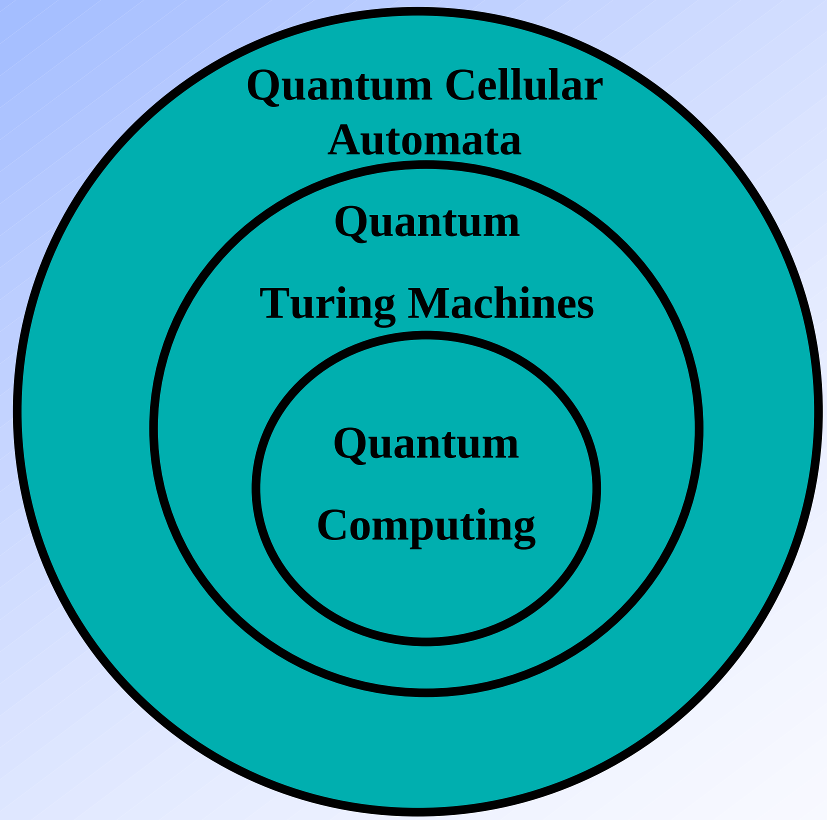
Outline

- Cellular Automata
- Quantum Cellular Automata (QCA)
- Relation to Lattice Field Theories
- Bosonic Quantum Cellular Automata
- Fermionic Quantum Cellular Automata
- Supersymmetric QCA
- Relation to String Bit Models
- Spin and Quantum Dot Cellular Automata
- Relation to Quantum Computing

Classical → Quantum





Cellular Automata

- Deterministic
- Discrete Time I, Space J, Target Space X
- Cell contents X(I,J)
- Update Rule f, reversible (Toffoli and Margolus (1990))

$$X(I+1,J) = f(X(I,J-1), X(I,J), X(I,J+1)) - X(I-1,J)$$

- Applications: Lattice gas, traffic models, classical computation (Wolfram 2003)

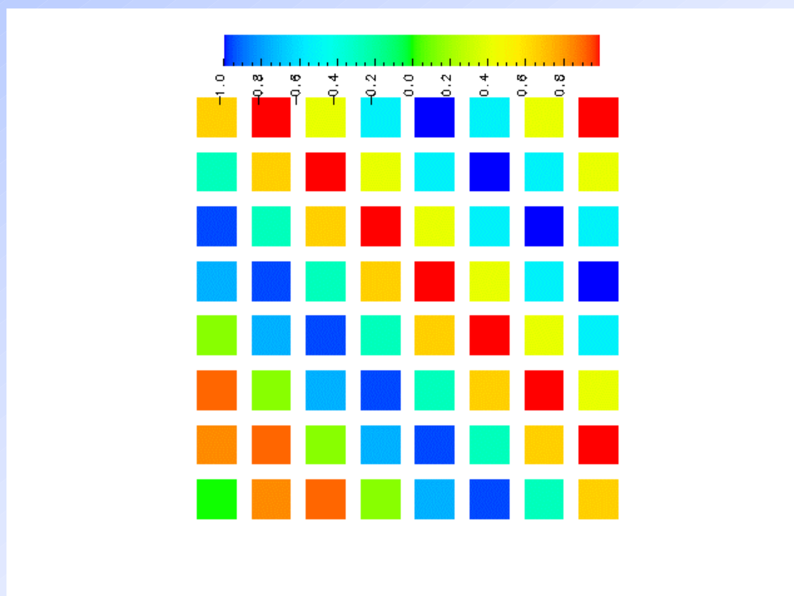
Cellular Automata Example

- Choose update function f

$$f(X(I,J-1),X(I,J),X(I,J+1)) = X(I,J-1)+X(I,J+1)$$

- update equation

$$X(I+1,J) = X(I,J-1)+X(I,J+1)-X(I-1,J)$$



Quantum Cellular Automata

- General QCA open problem. (Feynman (1982), Grossing and Zeilinger (1988), t'Hooft (1992), Meyer (1996))
- Non-deterministic: Applications quantum fluid, quantum computing
- Discrete time I and Quantum Mechanics (T.D.Lee (1983), Khorrami (1994),Jaroszkiewicz and Norton (1996))
- Discrete space J (spatial lattice).
- Target Space X, cell contents X(I,J) (lattice field)
- Action S = S{X(I,J)}
- Transition amplitude from Path Integral

$$K_{Z_k}(X(N,\cdot),N;X(0,\cdot)) = \prod_{I,J=1,0}^{N-1,M} \sum_{X(I,J)=0}^{k-1} e^{iS\{X\}}$$

Bosonic Quantum Cellular Automata

- Action

$$S = \sum_{I,J=0}^{N,M} \frac{1}{2}\left(\left(X(I+1,J) - X(I,J)\right)^2 - \left(X(I,J+1) - X(I,J)\right)^2\right)$$

- Equation of motion is update equation

$$X(I+1,J) = X(I,J-1) + X(I,J+1) - X(I-1,J)$$

- Solve by Fourier transformation

$$X(I,J) = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} X_n(I) e^{-2\pi i n J/M}$$

Bosonic Quantum Cellular Automata

- Update equation and action become:

$$X_n(I+1) = 2X_n(I) - W_n^2 X_n(I) - X_n(I-1)$$

$$S = \sum_{I=0}^{N-1} \left(\frac{1}{2}(X_n(I+1) - X_n(I))^2 - \frac{1}{2} W_n^{\,2} X_n^{\,2}(I)\right)$$

- with $W_n = 2\sin(\pi n / M) = 2\sin(a_0 \omega_n / 2)$

$$K(X(N,\cdot), N; X(0,\cdot), 0) = \prod_{I,J=1,0}^{N-1,M} \int dX(I,J) e^{iS\{X\}}$$

- Product of discrete time Harmonic Oscillators of the form:

$$K(X_N, N; X_0, 0) = \sqrt{\frac{\sin(a_0\omega_n)}{2\pi i \sin(a_0\omega_n N)}} \exp\left(\frac{i\sin(a_0\omega_n)}{2\sin(a_0\omega_n N)}\left((X_N^2 + X_0^2)\cos(a_0\omega_n N) - 2X_0 X_N\right)\right)$$

Discrete Target Space

- Cells can take k values

$$X \in \mathbb{Z}_k = \{0,1,\ldots,k-1\} \qquad X \approx X + k$$

- Action becomes

$$S = \left(\frac{k}{2\pi}\right)^2 \sum_{I,J=0}^{N-1,M} \cos(2\pi(X(I,J+1) - X(I,J))/k) - \cos(2\pi(X(I+1,J) - X(I,J))/k)$$

- Path integral

$$K_{Z_k}(X(N,\cdot), N; X(0,\cdot)) = \prod_{I,J=1,0}^{N-1,M} \sum_{X(I,J)=0}^{k-1} e^{iS\{X\}}$$

Discrete Target Space

- Special interest in $X \in \mathbb{Z}_2 = \{0,1\}$ Dead or alive
- Action

$$A = \left(\frac{1}{\pi}\right)^2 \sum_{I,J=0}^{N-1,M} \cos(\pi X(I,J+1))\cos(\pi X(I,J)) - \cos(\pi X(I+1,J))\cos(\pi X(I,J))$$

- Define Spin Variables $S_z(I,J) = \cos(\pi X(I,J))$

$$A = \left(\frac{1}{\pi}\right)^2 \sum_{I,J=0}^{N-1,M} S_z(I,J+1)S_z(I,J) - S_z(I+1,J)S_z(I,J)$$

- Path Integral related to Ising Model

$$K(S_z(N,\cdot),N;S_z(0,\cdot)) = \prod_{I,J=0}^{N-1,M} \sum_{S_z(I,J)=\pm 1} e^{iA\{S_z\}}$$

Fermionic Quantum Cellular Automata

- Fermionic cell contents $\theta(I, J)$

- Applications to condensed matter, string theory, quark lattice gauge theory

- Update equation

$$\theta(I+1, J) = f(\theta(I, J-1), \theta(I, J), \theta(I, J+1)) + \theta(I-1, J)$$

Example: Fermionic Cellular Automata

- Simplest update function

$$f(\theta(I,J-1),\theta(I,J),\theta(I,J+1))=\theta(I,J-1)-\theta(I,J+1)$$

- Update equation with solution $\theta(I,J)=\theta_R(I-J)$

$$\theta(I+1,J)=\theta(I,J-1)-\theta(I,J+1)+\theta(I-1,J)$$

- Another simple update function

$$\tilde{f}(\tilde{\theta}(I,J-1),\tilde{\theta}(I,J),\tilde{\theta}(I,J+1))=-\tilde{\theta}(I,J-1)+\tilde{\theta}(I,J+1)$$

- Update equation with solution $\tilde{\theta}(I,J)=\tilde{\theta}_L(I+J)$

$$\tilde{\theta}(I+1,J)=-\tilde{\theta}(I,J-1)+\tilde{\theta}(I,J+1)+\tilde{\theta}(I-1,J)$$

Fermionic Quantum Cellular Automaton

- Action:

$$S = \sum_{I,J=0}^{N-1,M-1} -i\frac{1}{2}\theta^*(I,J)(\theta(I+1,J)-\theta(I-1,J)) - i\frac{1}{2}\theta^*(I,J)(\theta(I,J+1)-\theta(I,J-1))$$

$$-i\frac{1}{2}\tilde{\theta}^*(I,J)(\tilde{\theta}(I+1,J)-\tilde{\theta}(I-1,J)) + i\frac{1}{2}\tilde{\theta}^*(I,J)(\tilde{\theta}(I,J+1)-\tilde{\theta}(I,J-1))$$

- Equation of motion yields update equation
- However fermion doubling problem with inverse propagator:

$$D_F(p_0, p_1) = \sin(p_0) + \sin(p_1)$$

- Zeros located $(0,0),(0,\pi),(\pi,0),(\pi,\pi)$

Fermionic Quantum Cellular Automata

- Wilson method to remove fermion doubling

$$S_r = \sum_{I,J=0}^{N-1,M-1} -i\frac{1}{2}(\theta^*(I,J)(\theta(I+1,J)-\theta(I-1,J)-r\tilde{\theta}(I+1,J)-r\tilde{\theta}(I-1,J)+2r\tilde{\theta}(I,J))$$

$$-i\frac{1}{2}(\theta^*(I,J)(\theta(I,J+1)-\theta(I,J-1)-r\tilde{\theta}(I,J+1)-r\tilde{\theta}(I,J-1)+2r\tilde{\theta}(I,J))$$

$$-i\frac{1}{2}(\tilde{\theta}^*(I,J)(\tilde{\theta}(I+1,J)-\tilde{\theta}(I-1,J)+r\theta(I+1,J)+r\theta(I-1,J)-2r\theta(I,J))$$

$$+i\frac{1}{2}(\tilde{\theta}^*(I,J)(\tilde{\theta}(I,J+1)-\tilde{\theta}(I,J-1)-r\theta(I,J+1)-r\theta(I,J-1)+2r\theta(I,J))$$

- Inverse propagator has a single zero at $p_0 = 0$

$$D_F(p_0) = \sin(p_0) + r(1-\cos(p_0)) = \sin(p_0) + 2r\sin^2(p_0/2)$$

Fermionic Quantum Cellular Automata

- Quantize by Grassmann path integral

$$K(\theta(N,\cdot),\tilde{\theta}(N,\cdot)N;\theta(0,\cdot),\tilde{\theta}(0,\cdot)) = \prod_{I,J=1,0}^{N-1,M} \int d\theta(I,J)d\tilde{\theta}(I,J)e^{iS_r\{\theta,\tilde{\theta}\}}$$

- Can Solve by Fourier transform and factor over frequencies

$$\theta(I,J) = \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} \theta_n(I) e^{-2\pi inJ/M}$$

Supersymmetric Quantum Cellular Automata

- Supersymmetry- interchange bosons and fermions.
- Used in String Models, Particle Physics, solvable Condensed Matter Models.
- Fermion doubling and supersymmetry implies boson doubling.
- Use first order formalism for bosons.

$$S=\sum_{I,J=0}^{N,M}\frac{1}{2}\left(P(I,J)^2-P(I,J)(X(I+1,J)-X(I-1,J))-L(I,J)^2+L(I,J)(X(I,J+1)-X(I,J-1))\right)$$

$$+\sum_{I,J=0}^{N-1,M-1}-i\frac{1}{2}\theta^*(I,J)(\theta(I+1,J)-\theta(I-1,J))-i\frac{1}{2}\theta^*(I,J)(\theta(I,J+1)-\theta(I,J-1))$$

Supersymmetric QCA

- Supersymmetry transformations

$$\delta X = i(\tilde{\varepsilon}\theta - \varepsilon\tilde{\theta})$$

$$\delta\theta = -\tilde{\varepsilon}\Big(\frac{1}{2}(X(I+1,J) - X(I-1,J) - X(I,J+1) + X(I,J-1)\Big)$$

$$\delta\tilde{\theta} = \varepsilon\Big(\frac{1}{2}(X(I+1,J) - X(I-1,J) + X(I,J+1) - X(I,J-1)\Big)$$

- To remove doublers replace:

$$X(I+1,J) - X(I-1,J) \Rightarrow X(I+1,J) - X(I-1,J) - r(X(I+1,J) + X(I-1,J) - 2X(I,J))$$

$$X(I,J+1) - X(I,J-1) \Rightarrow X(I,J+1) - X(I,J-1) - r(X(I,J+1) + X(I,J-1) - 2X(I,J))$$

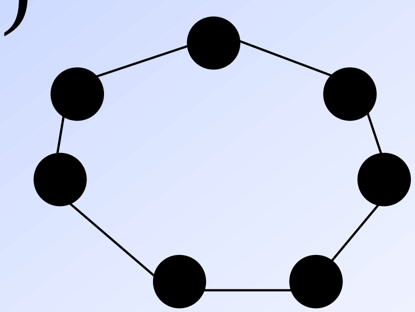
Supersymmetric QCA

- Quantize by path integral

$$K(X(N,\cdot),\theta(N,\cdot),\tilde{\theta}(N,\cdot)N; X(0,\cdot),\theta(0,\cdot),\tilde{\theta}(0,\cdot)) =$$

$$\prod_{I,J=1,0}^{N-1,M} \int dX(I,J)\,d\theta(I,J)\,d\tilde{\theta}(I,J)\,e^{iS_r\{X,\theta,\tilde{\theta}\}}$$

- Related to string bit models of Thorn and Bergman (1996)



Relation to String Models

- Continuum limit

$$S = \int d^2\sigma (\partial_\alpha X \partial^\alpha X - i\bar{\theta}\rho^\alpha \partial_\alpha \theta)$$

2+1 dimensional IIB superstring in lightcone gauge

- For Heterotic models 2D chiral fermions
- Use domain wall fermions to discretize

$$X(I,J,K) \to < J \,|\, X(I) \,|\, K > \to (X(t))_{JK}$$

- M-atrix theory of Banks, Fischler,Shenker and Susskind (1997)

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Spin Cellular Automata

- Spin valued cell contents $S_a[I,J] \quad a = x, y, z$

- Spin update equation

$$S_a(I+1,J) = \varepsilon_{abc} S_b(I,J) B_c + \varepsilon_{abc} S_b(I,J) S_c(I,J-1) + +\varepsilon_{abc} S_b(t,J) S_c(t,J+1) + S_a(I-1,J)$$

- Continuum time limit
  Bloch equation $a_0 \to 0 \qquad t = a_0 I$

$$\partial_t S_a(t,J) = \varepsilon_{abc} S_b(t,J) B_c + \varepsilon_{abc} S_b(t,J) S_c(t,J-1) + \varepsilon_{abc} S_b(t,J) S_c(t,J+1)$$

Spin Quantum Cellular Automata

- Quantize in Heisenberg picture

$$\partial_t S_a(t,J) = i[H, S_a(t,J)]$$

- Hamiltonian

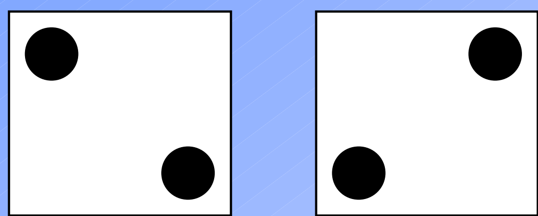
$$H = \sum_{J=1}^{M} S_a(t,J)S_a(t,J+1) + S_a(t,J)S_a(t,J-1) + S_a(t,J)B_a$$

- Transition function

$$K(S_z(T,\cdot); S_z(0,\cdot)) = < S_z(T,\cdot) \mid e^{-iTH} \mid S_z(0,\cdot) >$$

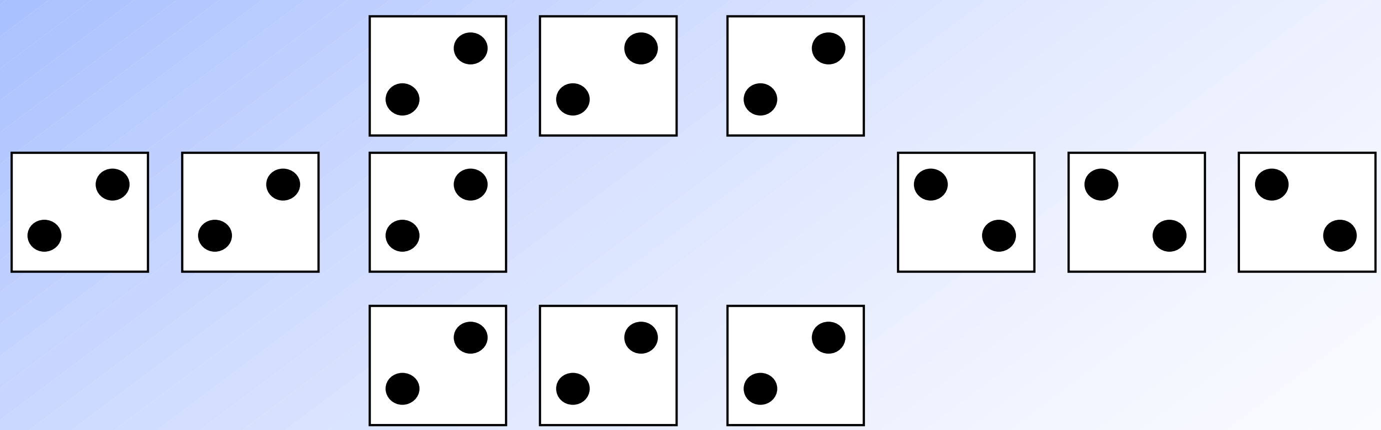
Quantum Dot Cellular Automata

- 4 -site 2-electron quantum dot (Tougaw, Lent, Porod (1993)



Nanoscale lithography in semiconductors

- Quantum Dot Cellular Automata inverter



Quantum Dot Cellular Automata

- Effectively a two state system with Hamiltonian (Adachi and Isawa (1998), Cole and Lusth (2001))

$$H_{eff} = \sum_{K=1}^{M} B_x S_K^x + \sum_{K=1}^{M} \gamma S_K^z S_{K+1}^z$$

- Ising model with transverse magnetic field. Effective exchange γ is negative , antiferromagnetic
- Described by Spin cellular automata

Relation to Quantum Computing

- **Standard Spin model of quantum computation. Spin control Hamiltonian for 1 and 2 qubit operations.**
- **Sufficient to build up any unitary operation (Barenco et al, Di Vincenzo (1995))**

$$H_{qc} = \sum_{K=1}^{M} (\alpha_x^K(t) S_K^x + \beta_y^K(t) S_K^y) + \sum_{K,L=1}^{M} \gamma^{KL}(t) S_K^z S_L^z$$

- **Standard Fermionic model of quantum computation. (Ortiz, Gubernatis, Knill, Laflamme (2000)**

$$H_{qc} = \sum_{K=1}^{M} (\alpha_K(t)\theta_K + \beta_K(t)\theta_K^\dagger) + \sum_{K,L=1}^{M} \gamma_{KL}(t)(\theta_K^\dagger \theta_L + \theta_L^\dagger \theta_K)$$

- **Related by Jordan-Wigner transformation and described by spin and fermionic cellular automata.**

Conclusions

- QCA can form a foundation for Quantum Computing but are poorly understood compared to Classical CA.
- Generalized to several types of Quantum Cellular Automata: Bosonic, Fermionic, Supersymmetric, Spin.
- Applications to Fundamental physics: String Bit Models.
- Application to Quantum Dot Cellular Automata and QC Control Hamiltonians.