Pure Spinors and Ramond-Ramond backgrounds (based on arXiv:0812.5074v1)

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– First Half –

Introduction

The difficulties in describing Ramond-Ramond backgrounds are well known and sometimes it's still said that it's not possible to describe them as a Conformal Field Theory. This is not true, although the CFT needed is different from the usual ones. To quantize the CFT in a ten dimensional super-Poincaré covariant way, the only option known so far is the Pure Spinor Formalism. The Green-Schwarz formalism can be used to describe these backgrounds classically, but it has quantization problems that will not be discussed here.

This talk is about interesting features of the pure spinor approach in the presence of Ramond-Ramond backgrounds. One doesn't have to have Ramond-Ramond backgrounds to use the pure spinor formalism. The formalism was originally written in a flat background, where it has manifest D=10 super-Poincaré invariance, but it was generalized to any background that satisfies the ten dimensional supergravity equations of motion, in particular $AdS_5 \times S^5$. $AdS_5 \times S_5$ is the most symmetrical Ramond-Ramond background and, in this background, the pure spinor action has manifest $PSU(2,2|4)$ symmetry.

In a flat background the theory is free and once the CFT is quantized, scaterring amplitudes can be immediately computed. In $AdS_5 \times S^5$ background the situation is more complicated since the two dimensional theory is interacting, but we can, at least, describe the rules to compute scattering amplitudes, even if actually performing the calculation is difficult. And it turns out that the rules are surprisingly different from the flat space case. Specifically, the rules for functionally integrating over the pure spinor variables are different from the flat space case and it is suspected that $AdS_5 \times S^5$ is special not because it is so symmetrical but because it is a Ramond-Ramond background. The $PSU(2,2|4)$ symmetry can't be smoothly connected to $D = 10$ super-Poincaré and it's possible that those features of the $AdS_5 \times S^5$ background also appear in other less symmetrical Ramond-Ramond backgrounds.

The first obvious difference in the $AdS_5 \times S^5$ background is the presence of a dimensionful parameter: the radius of AdS_5 which allows us to ask about certain properties of the theory at various values of this parameter. When this radius is small, this theory is well understood in terms of perturbative $\mathcal{N} = 4$, $D = 4$ super Yang-Mills although, until now, no one could quantize this theory in this regime to compare both theories. The quantization at large radius has been carried on and it is no more complicated than the flat space case, but the zero mode structure is quite different even for the simplest three-point tree level amplitude. Due to this difference, some non-renormalization theorems that are easily proved in a flat background haven't yet been proved in the $AdS_5 \times S^5$ background.

Review of the pure spinor formalism

In flat background, the type IIB pure spinor superstring has the following fields:

- x^m: 10 dimensional vector
- $\theta^\alpha, \hat{\theta}^{\hat{\alpha}}$: 10 dimensional fermionic spinors with same chirality
- $\lambda^\alpha, \hat{\lambda}^{\hat{\alpha}}$: 10 dimensional bosonic spinors with same chirality satisfying the pure spinor constarint $\lambda\gamma^m\lambda = \hat{\lambda}\gamma^m\hat{\lambda} = 0$.

Only 22 components of λ and $\hat{\lambda}$ are independent due to the pure spinor constraint. The action is very simple:

$$S = \int d^2z \left(\partial x^m \bar{\partial} x_m + p_\alpha \bar{\partial}\theta^\alpha + \hat{p}_{\hat{\alpha}} \partial \hat{\theta}^{\hat{\alpha}} + \omega_\alpha \bar{\partial}\lambda^\alpha + \hat{\omega}_{\hat{\alpha}} \partial \hat{\lambda}^{\hat{\alpha}} \right) \quad (1)$$

where $p, \hat{p}$ and $\omega, \hat{\omega}$ are the conjugate momenta to $\theta, \hat{\theta}$ and $\lambda, \hat{\lambda}$. Due to the pure spinor constraint, the $\omega, \hat{\omega}$ are defined up to the gauge transformation:

$$\delta\omega_\alpha = \Lambda_m \left(\gamma^m\lambda\right)_\alpha \qquad \delta\hat{\omega}_{\hat{\alpha}} = \hat{\Lambda}_m(\gamma^m\hat{\lambda})_{\hat{\alpha}} \quad (2)$$

There is no Majorana-Weyl solution for the pure spinor constraint, so the λ's parameterize the 11 dimensional complex space (after Wick-rotation):

$$\lambda^\alpha \in \frac{SO(10)}{U(5)} \times \mathbb{C} \tag{3}$$

where the first part picks, dinamically, a complex structure and the second part is the λ ghost number.

The idea in a curved background is to contract the indices with a general metric and introduce vielbeins. We also have to generalize the pure spinor action demanding that it is not only conformal invariant, but also BRST invariant. The BRST operator has the structure:

$$Q = \int dz\, \lambda^\alpha d_\alpha + \int d\bar{z}\, \hat{\lambda}^{\hat{\alpha}} \hat{d}_{\hat{\alpha}} \tag{4}$$

where, in the flat background, $d_\alpha = p_\alpha - \frac{1}{2}\left(\partial x^m + \frac{1}{4}\theta\gamma\partial\theta\right)(\gamma_m\theta)_\alpha$ and correspondingly for the anti-holomorphic part. In a curved background, the action will not be quadratic and it is more convenient to write the action directly in terms of d_α, where it is defined as the operator that (anti)commutes with the supersymmetry generator $\{\int q, d\} = 0$.

Ramond-Ramond backgrounds

In Ramond-Ramond background the action of the sigma model will present a coupling of the p's (written here in terms of d's):

$$S = \int d^2z\, G_{mn}(x,\theta)\partial x^m \partial x^n + F^{\alpha\hat{\beta}}(x,\theta) d_\alpha \hat{d}_{\hat{\beta}} + \cdots \tag{5}$$

where the lowest component of the superfield $F^{\alpha\hat{\beta}}$ is the Ramond-Ramond field strength. This means that if we work in a background in which F is invertible, we can integrate out the d's and write the whole action as a function of $x, \theta, \hat{\theta}, \lambda, \hat{\lambda}, \omega, \hat{\omega}$, but not $p, \hat{p}$. The integration over the zero modes of the p's will not be present, though the information of their existence remains on the fact that the θ^α's and $\hat{\theta}^\alpha$'s are not (anti)holomorphic anymore and the number of zero modes has changed. Something similar happens with the λ's

When calculating the three-point massless tree amplitude, the answer won't depend on where the vertex operators are located on the sphere and the only thing that has to be done is to integrate over the zero mode. In the massless supergravity case, the vertex operator is expressed in terms of a gauge superfield:

$$V = \lambda^\alpha \hat{\lambda}^{\hat{\beta}} A_{\alpha\hat{\beta}} \tag{6}$$

In a flat background, the terms with zero and one θ or $\hat{\theta}$ can be gauged away, writing the vertex operator schematically as $V \simeq \theta\hat{\theta}(NS\ fields) + \theta^2\hat{\theta}(gravitinos) + \theta\hat{\theta}^2(gravitinos) + \theta^2\hat{\theta}^2(RR\ fields)$. In a Ramond-Ramond background, it's no longer true that the vertex operator begins with the $\theta\hat{\theta}$ term, exactly due to the interaction between d and $\hat{d}$ in the action. So there are fields that will appear at order zero in the θ expansion.

The answer for the three point amplitude in a flat background is the trilinear coupling between the gravitons that comes by expanding the Einstein-Hilbert term:

$$\int d^{10}x\ (g\partial_m g\partial_n g) \tag{7}$$

This immediately tells how many θ's need to be pulled out to have a non-vanishing answer. Since each derivative has the same dimension as 2 θ's, the answer has to contain 10 θ's (5 θ's and 5 $\hat{\theta}$'s). The correct integration measure for the zero modes turns out to be:

$$\langle |(\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)|^2\rangle = 1 \tag{8}$$

This result can also be derived from functional integration:

$$\int d^{10}x \int d^{16}\theta\, d^{16}\hat{\theta} \int d^{11}\lambda\, d^{11}\hat{\lambda}\, (\lambda\hat{\lambda}A)^3 \tag{9}$$

Naively, this would give $\int d^{10}x \int d^5\theta\, d^5\hat{\theta}\, A^3\, (0^{11}\infty^{11})$. The correct way to regularize this prescription and obtain the previous result is to introduce non-minimal fields:

- $(\bar{\lambda}_\alpha, \bar{\omega}^\alpha)$
- $(\hat{\bar{\lambda}}_{\hat{\alpha}}, \hat{\bar{\omega}}^{\hat{\alpha}})$
- $(\bar{r}_\alpha, \bar{s}^\alpha)$

- $(\hat{\bar{r}}_{\hat{\alpha}}, \hat{\bar{s}}^{\hat{\alpha}})$

such that $\bar{\lambda}\gamma^m\bar{\lambda} = r\gamma^m\bar{\lambda} = \hat{\bar{\lambda}}\gamma^m\hat{\bar{\lambda}} = \hat{r}\gamma^m\hat{\bar{\lambda}} = 0$. and introduce the regulator $\mathcal{N} = 1 + \{Q, \Lambda\} = e^{-\{Q,\chi\}} = e^{-(\lambda\bar{\lambda} + \hat{\lambda}\hat{\bar{\lambda}} + r\theta + \bar{r}\bar{\theta})}$ in the functional integral. Those non-minimal fields act as a contractible pair and, to not affect the cohomology of the BRST operator, Q has to be modified as:

$$Q = \int dz\, \lambda d + \bar{\omega} r \tag{10}$$

In flat space, it's natural to interpret $\bar{\lambda}$ as the complex conjugate to λ and $\chi = \theta\bar{\lambda} + \hat{\theta}\hat{\bar{\lambda}}$ introduces a gaussian regulator to the functional integral. $\mathcal{N}$ does nothing to the amplitude since it is almost BRST trivial but when we integrate also over the non-minimal variables, we get a well defined expression.

To calculate loop amplitudes it's necessary to introduce a b ghost. Since the pure spinor doesn't come from a reparametrization invariant action, the b ghost is not an original field from the action but a composite field, defined such that $\{Q, b\} = T$. The expression for the b ghost in a flat background is

$$b_{flat} = \frac{2\bar{\lambda}_\alpha \gamma^m (\gamma_m d)^\alpha - \bar{\lambda}_\alpha N_{mn} (\gamma^{mn} \partial\theta)^\alpha}{4(\bar{\lambda}\lambda)} + \cdots \tag{11}$$

In a Ramond-Ramond background, things will be simpler as the $AdS_5 \times S^5$ case prototypically shows (the plane wave case is worked out in the paper)

$AdS_5 \times S^5$ **background**

The simplification that occurs in the $AdS_5 \times S^5$ background can be already seen by taking the supersymmetry variation of the lagrangian. In a flat background $q\mathcal{L}_{flat} = \partial(\cdots)$ where the total derivative comes from the Wess-Zumino term of the action. This is directly related to the fact that the scattering amplitude is only spacetime supersymmetric on-shell. On the other hand, in an $AdS_5 \times S^5$ background the lagrangian is $PSU(2,2|4)$ invariant without total derivatives. This means that we must be able to write vertex operators and integration measure that are $PSU(2,2|4)$ invariant, not only up to BRST exact terms. This doesn't mean that it is known how to compute off-shell amplitudes with string theory, but it implies at least that the prescription is $PSU(2,2|4)$ invariant for any state.

The action of the pure spinor superstring in $AdS_5 \times S^5$ is constructed as a sigma model for the coset $PSU(2,2|4)/(SO(4,1) \times SO(5)$ as originally done by Metsaev-Tseytlin [hep-th/9805028] in the context of GS superstring. $PSU(2,2|4)$ has 30 bosonic generators and 32 fermionic generators:

- P_m: translation generators
- $J_{[ab]}$: "lorentz" generators for $SO(4,1)$ and $SO(5)$ - *coset out.*
- Q_α, $\hat{Q}_{\hat{\alpha}}$: "supersymmetry" generators

The action will be written in terms of the left-invariant currents $J^A = (g^{-1}dg)^A$ where the index takes value in the Lie algebra $psu(2,2|4)$: $A = ([ab], a, \alpha, \hat{\alpha})$. The global $PSU(2,2|4)$ transformations will act as a left transformation $g \to \Lambda g$. The action, after integrating out d and $\hat{d}$, is:

$$S = \int d^2z\,[\frac{1}{2}(\eta_{ab}J^a\bar{J}^b + \eta_{\alpha\hat{\beta}}J^\alpha\bar{J}^{\hat{\beta}} + \eta_{\alpha\hat{\beta}}\bar{J}^\alpha J^{\hat{\beta}}) - \frac{1}{4}\eta_{\alpha\hat{\beta}}(J^\alpha\bar{J}^{\hat{\beta}} - \bar{J}^\alpha J^{\hat{\beta}}) + (-\omega_\alpha\bar{\nabla}\lambda^\alpha + \hat{\omega}_{\hat{\alpha}}\nabla\hat{\lambda}^{\hat{\alpha}} - \eta_{[ab][cd]}N^{ab}\hat{N}^{cd})] \quad (12)$$

where $\nabla = \partial + J_{[ab]}\omega^{[ab]}$. Different from the flat space case, there exists a metric element which contracts non-hatted with hatted spinor indices $\eta_{\alpha\hat{\alpha}} = (\gamma^{01234})_{\alpha\hat{\alpha}}$. In flat space, this would break the ten dimensional Poincaré invariance, but here it is consistent with all the symmetries of the action. Although strange, it can be shown that this is the only BRST invariant action where, using the equations of motion to $d, \hat{d}$, the BRST operator is:

$$Q = \int dz\,\lambda^\alpha d_\alpha + \int d\bar{z}\,\hat{\lambda}^{\hat{\alpha}}\hat{d}_{\hat{\alpha}} = \int dz\,\eta_{\alpha\hat{\alpha}}\lambda^\alpha J^{\hat{\alpha}} + \int d\bar{z}\,\eta_{\alpha\hat{\alpha}}\hat{\lambda}^{\hat{\alpha}}\hat{J}_\alpha \quad (13)$$

And the field g in the action transforms as:

$$Qg = g(\lambda^\alpha T_\alpha + \hat{\alpha}T_{\hat{\alpha}}) \quad (14)$$

where $T, \hat{T}$ are the $PSU(2,2|4)$ matrices corresponding to the supersymmetry generators $Q, \hat{Q}$. – **Second Half –**

Vertex Operator

We will work out here only the simplest case, but the general massless case has been worked out by Mikhailov [arXiv:0903.5022]. The simplest vertex operator can be constructed as in bosonic string by taking the BRST variation of the lagrangian:

$$Q\mathcal{L} = \partial \bar{f} + \bar{\partial} f \Rightarrow Q\bar{f} = \bar{\partial} v, \quad \bar{Q} f = -\partial v \tag{15}$$

where v is the unintegrated vertex operator at zero momentum. For the bosonic string, this gives $v = c\bar{c}\partial x^m \bar{\partial} x_m$, while for the flat pure spinor superstring this gives $v = (\lambda\gamma^m\theta)(\hat{\lambda}\gamma_m\hat{\theta})$. This last expression is only supersymmetric invariant up to a BRST trivial term and this is again related to the fact that the action shifts by a total derivative.

For the $AdS_5 \times S^5$, the unintegrated vertex operator related to the lagrangian is no longer the product of two fermions, but has a part that is purely constructed out of λ's:

$$v = \eta_{\alpha\hat{\alpha}} \lambda^\alpha \hat{\lambda}^{\hat{\alpha}} \tag{16}$$

This operator is automatically in the cohomology since it is related to an integrated vertex operator. This is not in the cohomology of the flat background BRST operator since $(\eta\lambda\hat{\lambda}) = Q(\eta\theta\hat{\lambda})$. The presence of this term in the cohomology will simplify the rules to compute functional integration.

In an $AdS_5 \times S^5$ background, it is natural to associate $\bar{\lambda} = \eta_{\alpha\hat{\alpha}}\hat{\lambda}^{\hat{\alpha}}$ as the complex conjugate to λ^α. This means that on a two-dimensional Euclidean worldsheet, the left-movers and right-movers are complex conjugates and the action is real. Note that in a flat background, it is more natural to associate the non-minimal variable $\bar{\lambda}_\alpha$ as the complex conjugate to λ^α, meaning that if the product $\bar{\lambda}\lambda$ is zero then all the components of λ is zero. In a flat background, this association means that the regulator provides a Gaussian cutoff. However, in an $AdS_5 \times S^5$ background, there is no need to introduce non-minimal variables or regulators, and the right-moving pure spinor ghosts can be associated with the complex conjugates of the left-moving pure spinor ghosts.

Having made this association, the vertex operator $\eta\lambda\hat{\lambda}$ only makes sense if we remove the point where $\lambda^\alpha = 0$. To compute the three-point amplitude in $AdS_5 \times S^5$, we can start with the naive definition:

$$\langle (\lambda\hat{\lambda}A)^2 \rangle = \int d^{10}x \int d^{11}\lambda \, d^{11}\bar{\lambda} \; \delta(\lambda\hat{\lambda} - c) \int d^{16}\theta \, d^{16}\hat{\theta} \; E(x,\theta,\hat{\theta})(\lambda\hat{\lambda}A)^3 \tag{17}$$

If we integrate over $\int d^{16}\theta \, d^{16}\hat{\theta}$ in flat background, this would give zero by dimensional analysis. The typical term would be $\int g\partial^a g \partial^b g$ with $a + b = 13$ and there is no way to contract the indices to get a non-vanishing expression since, by momentum conservation, $k_i \cdot k_j = 0$. In $AdS_5 \times S^5$, however, we have the dimensionful radius parameter R that allows us to write $\int (g\partial g\partial g)/R^{11}$.

There are many checks that this is the correct answer:

- The prescription is $PSU(2,2|4)$ invariant. For 3-point massless tree amplitudes, $PSU(2,2|4)$ invariance uniquely determines the amplitude up to an overall constant.
- The b-ghost, defined such that $\{Q,b\} = T = \eta_{ab}J^aJ^b/2 + \eta_{\alpha\hat{\alpha}}J^\alpha J^{\hat{\alpha}} + \omega_\alpha \nabla\lambda^\alpha$ is the obvious generalization of the first term in the flat space case:

$$b = (\eta\lambda\hat{\lambda})^{-1}\hat{\lambda}^{\hat{\alpha}}[\frac{1}{2}\gamma_{a\hat{\alpha}\hat{\beta}}J^aJ^{\hat{\beta}} + \frac{1}{4}(\gamma_{ab})_{\hat{\alpha}}^{\hat{\beta}}\eta_{\beta\hat{\beta}}N^{ab}J^\beta + \frac{1}{4}\eta_{\alpha\hat{\alpha}}J_{gh}J^\alpha] \tag{18}$$

Of course, the definitive way to check if this is the correct prescription is to actually calculate scattering amplitudes, but this hasn't been done so far.

References