Supersymmetric Wilson loops from weak to strong coupling

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Abstract

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1 Wilson loops in $\mathcal{N}=4$ SYM

In these notes I will give an overview of several results concerning a certain family of supersymmetric Wilson loop operators in $\mathcal{N}=4$ SYM theory and their string theory duals in $AdS_5 \times S^5$. These operators were introduced in [1–3] and further studied in [4, 5]. I will take the gauge group to be $SU(N)$, and I will mostly consider the gauge theory to be defined on Euclidean $\mathbb{R}^4$ (but by conformal invariance we may also do a conformal transformation to e.g. S^4 or $\mathbb{R} \times S^3$).

One motivation to study such Wilson loops is for example the general classification of supersymmetric non-local operators in SUSY gauge theories. Another interesting motivation is that in certain cases supersymmetric Wilson loops may provide examples of exactly calculable physical observables, interpolating between weak and strong coupling. In particular one can perform this way precise and non-trivial tests of the AdS/CFT correspondence.

In any gauge theory one can define the standard Wilson loop as the holonomy of the gauge field around a closed loop $\mathcal{C}$. In a supersymmetric gauge theory, it is natural to generalize this definition by adding couplings to the scalar fields which belong to the same multiplet of the gauge field. In the $\mathcal{N}=4$ theory, we couple the Wilson loop to the six real scalars as

$$W_R(\mathcal{C}) = \mathrm{tr}_R\, \mathrm{Pexp} \oint_C ds \left(A_\mu \dot{x}^\mu + i\Theta^I(s)\Phi^I \right) , \qquad I = 1, \ldots, 6, \tag{1.1}$$

where R is a choice of representation of the gauge group, and $\Theta^I(s)$ is, at this point, an arbitrary curve on $\mathbb{R}^6$. A similar definition applies to theories with lower supersymmetries. For example, in an $\mathcal{N}=2$ theory, one can couple the loop operator to the two real scalars of the $\mathcal{N}=2$ vector multiplet.

Consider now the supersymmetry variation of the operator $W_R(\mathcal{C})$. By "supersymmetry", we mean here the full 32 fermionic symmetries of the superconformal group $PSU(2,2|4)$, namely the 16 Poincare plus the 16 superconformal supercharges. The variation of the gauge field and scalars can be written as

$$\delta A_\mu = \Psi\gamma_\mu\epsilon(x)\,, \qquad \delta\Phi^I = \Psi\rho^I\epsilon(x)\,. \tag{1.2}$$

Here Ψ is the 16-component gaugino, γ_μ, ρ^I are Dirac matrices and $\epsilon(x)$ is the conformal Killing spinor

$$\epsilon(x) = \epsilon_0 + x^\mu\gamma_\mu\epsilon_1\,, \tag{1.3}$$

where ϵ_0 and ϵ_1 are 16-components constant spinors corresponding respectively to the Poincare and superconformal supercharges. The variation of the Wilson loop operator gives then

$$\delta W_R(\mathcal{C}) \propto \left(\dot{x}^\mu\gamma_\mu + i\Theta^I(s)\rho^I\right)\epsilon(x)\,. \tag{1.4}$$

From the form of this variation, one can see that a necessary condition for supersymmetry is that Θ^I is a unit vector

$$\Theta^I\Theta^I = 1 \tag{1.5}$$

i.e. it describes a curve on S^5. Indeed in this case the operator acting on $\epsilon(x)$ squares to zero (for simplicity, we have chosen a loop parameterization such that $|\dot{x}|=1$), and can be used to project out half of the components of the spinor. However, in general the operator will only be "locally BPS", since the projector, and hence the preserved spinor, depends on the point along the loop. The Wilson loop will be truly globally supersymmetric only if it preserves the same fraction of supersymmetry independently of the point along the loop. This can be achieved by suitably choosing the couple $\left(x^\mu(s), \Theta^I(s)\right)$ defining the operator.

2 Example: The 1/2 BPS circle

A simple and well-known example of supersymmetric Wilson loop can be obtained as follows: take the loop $x^\mu(s)$ to be a circle, e.g. on the (x^1, x^2) plane, and take Θ^I to be a constant unit vector, e.g. $\Theta^I = (1,0,\dots,0)$. By conformal invariance, we can take the radius of the circle to be one. The corresponding operator couples to one of the scalar fields with constant strength

$$W_R = \mathrm{tr}_R\,\mathrm{Pexp}\oint ds(A_\mu\dot{x}^\mu + i\phi^1)\,. \tag{2.1}$$

The supersymmetry variation (1.4) vanishes provided ϵ_0 and ϵ_1 are related by

$$i\rho^1\epsilon_0 = \gamma_{12}\epsilon_1. \tag{2.2}$$

This is a single equation relating ϵ_0 and ϵ_1, hence the operator preserves 16 linear combinations of Poincare and superconformal supercharges, i.e. half of the supersymmetries of the vacuum [1].

As conjectured by Erickson, Semenoff and Zarembo [6] and Drukker and Gross [7], and recently proved by Pestun [8], the expectation value of this operator is exactly captured by the following Hermitian Gaussian matrix model

$$\langle W_R \rangle = \langle \mathrm{tr}_R \, e^X \rangle_{\mathrm{m.m.}} = \frac{1}{\mathcal{Z}} \int DX \, e^{-\frac{2}{g^2} \mathrm{tr} \, X^2} \, \mathrm{tr}_R \, e^X \,, \tag{2.3}$$

where X is an hermitian $N \times N$ matrix, g is the Yang-Mills coupling constant and $\mathcal{Z}$ is the matrix model partition function. As written above, the result applies to $U(N)$ gauge group. For general gauge group G, the integral is taken over the Lie algebra of G.

The prove of Pestun is based on conformally mapping the theory to S^4 and then studying localization of the path integral on supersymmetric configurations (by conformal invariance, the expectation value of W_R on $\mathbb{R}^4$ around the trivial conformal vacuum $< \phi^I >= 0$ is equal to the expectation value of W_R on S^4). In short, the idea of the proof is as follows: if Q is one of the supersymmetries preserved by W_R, one can deform the Yang-Mills action by

$$S_{YM} \to S_{YM} + tQV \tag{2.4}$$

where t is a parameter and V is a suitably chosen functional which is Q^2-invariant (Q does not square to zero, but it squares to a bosonic symmetry). Then by standard arguments one can see that the expectation value of Q-closed observables, such as W_R, is t-independent. In particular one can take the limit $t \to \infty$, in which the path integral localizes to the configurations solving $QV = 0$, weighted by the one-loop determinant for fluctuations around the localization locus. By studying the explicit action of the supersymmetry Q, one can argue that the localization locus is given by $\phi^1 = const$ and all other fields set to zero. The one-loop determinant turns out to be trivial in the $\mathcal{N} = 4$ theory, and one directly gets that the whole path-integral localizes to the above matrix model (the Gaussian potential comes from the conformal coupling of the scalars to the S^4 curvature). The same 1/2 BPS operator (2.1) can be studied in $\mathcal{N} = 2$ theories (in this case ϕ^1 is one of the two real scalars of the $\mathcal{N} = 2$ vector multiplet). Pestun's localization calculation applies to this situation as well, but in this case the one-loop determinant is non-trivial and there is also a non-trivial instanton contribution [2]. Then one gets the considerably more complicated matrix model

$$\langle W_R \rangle_{\mathcal{N}=2} = \frac{1}{\mathcal{Z}} \int DX \, e^{-\frac{2}{g^2} \mathrm{tr} \, X^2} Z_{\text{1-loop}}(X) |Z_{\text{inst}}(X)|^2 \, \mathrm{tr}_R \, e^X \,, \tag{2.5}$$

[1] One may also consider a straight line coupled to a single scalar in the same way. In this case the operator preserves separately 8 Poincare and 8 superconformal supersymmetries.

[2] The instantons are point-like instanton/anti-instantons localized at the north/south pole of the S^4. Their contribution turns out to be trivial in the $\mathcal{N} = 4$ theory.

where $Z_{\text{inst}}(X)$ is Nekrasov's instanton partition function.

Going back to the $\mathcal{N}=4$ theory, since the matrix model is Gaussian, it can be easily solved exactly. In particular, for the case of the fundamental representation and taking the planar limit, one readily gets [3]

$$\langle W_F \rangle \overset{N\to\infty}{=} \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) = \begin{cases} 1+\frac{\lambda}{8}+\frac{\lambda^2}{192}+\dots & \lambda \ll 1 \\ \sqrt{\frac{2}{\pi}}\frac{e^{\sqrt{\lambda}}}{\lambda^{\frac{3}{4}}}(1-\frac{3}{8\sqrt{\lambda}}+\dots) & \lambda \gg 1 \end{cases}, \qquad (2.6)$$

where $I_1(x)$ is a modified Bessel function of the first kind, and $\lambda = g^2 N$ is the 't Hooft coupling. As displayed in the equation above, this result nicely interpolates between a perturbative Feynman diagram expansion and a strong coupling expansion at large λ. All currently available results, both from perturbation theory and from string theory, agree with the exact matrix model prediction.

A natural question is now: can we find interesting generalizations to operators preserving less supersymmetries which may still be exactly calculable?

3 Zarembo's supersymmetric Wilson loops

An earlier general construction of supersymmetric Wilson loops is due to Zarembo [9]. The construction goes as follows: take an *arbitrary* loop on $\mathbb{R}^4$, and pick 4 out of the 6 scalars, ϕ_μ, $\mu = 1, \dots, 4$. Then we can define the following loop operator by coupling ϕ_μ to the tangent vector to the loop

$$W_R(\mathcal{C}) = \mathrm{tr}_R \,\mathrm{Pexp} \oint_{\mathcal{C}} ds \dot{x}^\mu \left(A_\mu + i\phi_\mu\right) . \qquad (3.1)$$

By studying the supersymmetry variation, one finds that for arbitrary loop this operator preserves one Poincare supercharge. Supersymmetry can be enhanced for loops of special shape or lying in lower dimensional subspaces of $\mathbb{R}^4$. It turns out that due to the preserved Poincare supersymmetry, all these operators have trivial expectation value

$$\langle W_R(\mathcal{C}) \rangle = 1 \qquad (3.2)$$

to all orders in g, N. This has been successfully checked both on the gauge theory and string theory side.

Note that the 1/2 BPS circle discussed earlier is not included in Zarembo's loops. We would like to find a new family of supersymmetric loop operators which have non-trivial vev's and include the 1/2 BPS circle as a special case. This is discussed in the next section.

[3] It is also easy to get the answer at finite N, which turns out to be $\langle W_F \rangle = \frac{1}{N} L^1_{N-1}(-g^2/4) e^{g^2/8}$, where $L^1_{N-1}(x)$ is a Laguerre polynomial.

4 The loops on S^3

Our construction of supersymmetric Wilson loops on S^3 [1, 3] goes as follows. Take an S^3 subspace of $\mathbb{R}^4$ defined by $x^\mu x^\mu = 1$ (for simplicity we can take the sphere to have unit radius, the radius does not matter by conformal invariance), and an *arbitrary* loop $x^\mu(s)$ on S^3. Then pick 3 out of the 6 scalars, $\Phi^i = (\Phi^1, \Phi^2, \Phi^3)$ and define the following loop operator

$$W_R(\mathcal{C}) = \mathrm{tr}_R \,\mathrm{Pexp} \oint_{\mathcal{C}} ds \left(A_\mu \dot{x}^\mu + i\sigma^i_{\mu\nu} x^\mu \dot{x}^\nu \Phi^i \right) . \tag{4.1}$$

Here $\sigma^i_{\mu\nu}$ are 't Hooft symbols, or equivalently the components of the S^3 left-invariant one-forms in Cartesian coordinates. They can be taken to be

$$\sigma^i_{jk} = \epsilon_{ijk} , \qquad \sigma^i_{4j} = -\sigma^i_{j4} = \delta^i_j . \tag{4.2}$$

Note that the construction associates to a given loop on S^3 a "dual" loop $\Theta^i = \sigma^i_{\mu\nu} x^\mu \dot{x}^\nu$ on a S^2 in the scalar field space.

Studying the supersymmetry variation (1.4), one finds that the loop dependence drops out, and for arbitrary curve on S^3 the operator (4.1) preserves 2 linear combinations of Poincare and superconformal supercharges, i.e. it is 1/16 BPS. Since the shape of the loop does not matter for supersymmetry, one may also consider several loops on S^3, and the combined system will still be 1/16 BPS.

It is easy to realize that the 1/2 BPS circular loop is contained in this general family as a special case: simply take the loop to be a great circle of the S^3, then $\sigma^i_{\mu\nu} x^\mu \dot{x}^\nu = const$ and one obtains the operator (2.1) discussed before.

As the 1/2 BPS circle, all these operators have non-trivial expectation values, and it is natural to ask whether they may be exactly calculable. While the most general case is not well understood yet, there is a special (infinite) subfamily of loops for which we have conjectured exact results in terms of 2d YM, as will be discussed next.

5 The 1/8 BPS loops on S^2 and 2d YM

Take a great S^2 of the S^3 defined above, e.g.

$$x^\mu x^\mu = 1 , \quad x^4 = 0 . \tag{5.1}$$

On this S^2 we can put loops of arbitrary shapes and couple them to the 3 scalars as prescribed in the previous section. In this case the Wilson loop (4.1) may be written as (specializing to $x^4 = 0$)

$$W_R = \mathrm{tr}_R \,\mathrm{Pexp} \oint ds \left(A_i \dot{x}^i + i(\vec{x} \times \dot{\vec{x}}) \cdot \vec{\Phi} \right) , \qquad i = 1, 2, 3 \tag{5.2}$$

We see that the vector $\vec{\Theta} = \vec{x} \times \dot{\vec{x}}$ coupling to the 3 scalars may be thought of as the angular momentum of a point particle running along the loop $x^i(s)$. From this point of view, it is clear that a great circle corresponds to the 1/2 BPS loop, since for a great circle the angular momentum is constant and orthogonal to the plane of the circle.

The analysis of the supersymmetry reveals that for arbitrary loops on this great $S^2 \subset S^3$ SUSY is doubled and the loops are in general 1/8 BPS. Loops of special shape may have enhanced supersymmetry: for example a latitude circle is a 1/4 BPS operator for general latitude angle (and becomes 1/2 BPS at the equator).

Motivated by several evidences both at weak coupling and from string theory, we conjectured that the expectation values and correlation functions of these operators on S^2 may be computed exactly in terms of the standard Wilson loops of 2d Yang-Mills on S^2, or more precisely of a peculiar "perturbative" truncation of 2d YM which consists in dropping the contributions of the non-trivial 2d instantons on S^2

$$\langle W_R(\mathcal{C}) \rangle_{4d} = \langle \mathrm{tr}_R \, P e^{\oint_{\mathcal{C}} A_{2d}} \rangle^{0\text{-inst}}_{YM_2} . \tag{5.3}$$

The "zero-instanton" sector of 2d YM is in turn related to a Hermitian Gaussian matrix model with coupling constant rescaled by an area-dependent factor

$$\langle \mathrm{tr}_R \, P e^{\oint_{\mathcal{C}} A_{2d}} \rangle^{0\text{-inst}}_{YM_2} = \frac{1}{\mathcal{Z}} \int DX \, e^{-\frac{2N}{\lambda'} \mathrm{tr}\, X^2} \; \mathrm{tr}_R \, e^X , \qquad \lambda' = \frac{4A_1 A_2}{A^2} \lambda \tag{5.4}$$

where $A_{1,2}$ are the areas of the two regions of S^2 singled out by the loop, and $A = A_1 + A_2$. The relation between the "zero-instanton" sector of 2d YM and the Hermitian matrix model was clarified in a series of papers by Bassetto et al, see e.g. [10, 11]. In general, 2d YM exact results may be written as integrals over the group manifold of the gauge group G. Dropping the non-trivial instantons corresponds to neglecting the global structure of the group G and approximating it with its Lie algebra, i.e. the Hermitian matrices in the case of $G = U(N)$. The simple area dependence is as expected in 2d YM because of the well-known invariance under area preserving diffeomorphisms, but it is a very non-trivial fact from the point of view of the 4d $\mathcal{N} = 4$ SYM theory.

Similarly, one can obtain exact results for correlators of several Wilson loops on S^2. In this case the "zero-instanton" sector reduces to a Gaussian multi-matrix model

$$\langle W_{R_1}(\mathcal{C}_1) W_{R_2}(\mathcal{C}_1) \cdots W_{R_k}(\mathcal{C}_k) \rangle = \langle \mathrm{tr}_{R_1} e^{X_1} \, \mathrm{tr}_{R_2} e^{X_2} \cdots \mathrm{tr}_{R_k} e^{X_k} \rangle_{\text{k-m.m.}} , \tag{5.5}$$

where the potential of the k-matrix model is a quadratic form in the $X_1, \ldots, X_k$ with area dependent coefficients.

As a consistency check of the conjecture (5.3), note that for the case of the 1/2 BPS great circle, for which $A_1 = A_2 = A/2$, we fall back to the matrix model (2.3), which has been proved to be correct.

A first simple evidence in favor of the conjecture comes from computing the expectation value of W_R in perturbation theory. At first order in λ, one has to compute a simple Feynman diagram in which a combined gauge field-scalar propagator is exchanged between two points along the loop, schematically

$$\langle W_R(\mathcal{C})\rangle = 1 + \lambda \int_{\mathcal{C}} ds_1 ds_2\, \dot{x}_1^i \dot{x}_2^j G_{ij} + \dots\,, \tag{5.6}$$

where G_{ij} is the combined gauge field-scalar propagator. Surprisingly, despite coming from 4d fields, this propagator turns out to be a Green's function for a gauge field on S^2. Hence, at this order of perturbation theory, the calculation is equivalent to a standard Wilson loop calculation in 2d YM and one easily gets, e.g. by Stokes theorem (here for concreteness we specialize to the case of fundamental representation)

$$\langle W_F(\mathcal{C})\rangle = 1 + \frac{4A_1A_2}{A^2}\frac{\lambda}{8} + \dots \tag{5.7}$$

in agreement with the perturbative expansion of the matrix model (5.4), see eq. (2.6). At next order of perturbation theory the calculation is considerably more complicated, as one encounters diagrams involving the interaction vertices of the $\mathcal{N}=4$ SYM lagrangian. However, the explicit calculation of [12], [13] shows again agreement with the 2dYM/matrix model conjecture (5.3)-(5.4).

A result in support of the conjecture has also been recently obtained by Pestun [14] using the localization framework. Generalizing the work on the 1/2 BPS circle to the case of our 1/8 BPS loops on S^2, he finds that in this case the path-integral localizes not directly to a matrix model, but to a 2d theory on S^2 which can be argued to be perturbatively equivalent to 2d YM. The one-loop determinant (as well as possible instanton corrections) were not computed in [14], so the conjecture was not yet proved. However further non-trivial evidence in favor of our proposal comes from string theory in $AdS_5 \times S^5$, where all available results are in agreement with the strong coupling limit of the conjecture (5.3)-(5.4).

6 Wilson loops in AdS

According to the well understood AdS/CFT dictionary, a Wilson loop in the fundamental representation along a curve $\mathcal{C}$ is described in AdS space by a fundamental string worldsheet ending at the boundary of AdS on $\mathcal{C}$ [4]. Let us take the $AdS_5 \times S^5$ metric as (we set the radius to one)

$$ds^2 = \frac{1}{z^2}\left(dz^2 + dx^\mu dx^\mu\right) + d\Omega_5^2\,, \tag{6.1}$$

[4]In the planar limit, one considers a surface with topology of a disk. Non-planar corrections correspond to higher genus worldsheets with one boundary.

where $d\Omega_5^2$ is the S^5 metric. The boundary of AdS_5 is at $z = 0$ and it is the $\mathbb{R}^4$ on which the gauge theory is defined.

The string worldsheet dual to the Wilson loop is a minimal surface, i.e. a solution to the classical equations of motion of the string σ-model on $AdS_5 \times S^5$, with boundary conditions

$$X^M(\tau, \sigma)|_{z=0} = \left(x^\mu(\tau), \Theta^I(\tau)\right) , \tag{6.2}$$

where X^M collectively denotes the $AdS_5 \times S^5$ coordinates, and τ, σ are the worldsheet coordinates. Here $x^\mu(\tau)$ and $\Theta^I(\tau)$ are identified respectively with the loop $\mathcal{C}$ on $\mathbb{R}^4$ and the curve on S^5 which couples to the scalars, as given in the definition of the Wilson operator (1.1).

The AdS/CFT prediction for the Wilson loop expectation value is obtained by computing the string partition function around this classical solution. At leading order in the α' expansion, which is equivalent to a $1/\sqrt{\lambda}$ expansion on the gauge theory side, this is just given by the exponential of the string on-shell action (i.e. the worldsheet area), suitably regularized to subtract the divergence due to the infinite length of the string (since it reaches the boundary)

$$\langle W \rangle = \mathcal{Z}_{F1} \overset{\alpha' \to 0}{\simeq} e^{-S_{F1}^{\text{reg}}} . \tag{6.3}$$

As an example, the solution to the minimal area problem for the 1/2 BPS circular loop is particularly simple. Taking the loop to be a unit circle on the x_1, x_2 plane, the dual string worldsheet in AdS is given by

$$x_1^2 + x_2^2 + z^2 = 1 , \qquad x_3 = x_4 = 0 , \tag{6.4}$$

while the worldsheet is taken to be pointlike on S^5, corresponding to the fact that the gauge theory operator couples to a single scalar field, see eq. (2.1). Geometrically, this surface is an AdS_2 embedded in $AdS_5 \times S^5$. Its regularized area can be easily computed to be (here we use the relation $\alpha' = 1/\sqrt{\lambda}$)

$$S_{F1}^{\text{reg}} = \frac{\sqrt{\lambda}}{4\pi} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \partial_\alpha X^M \partial_\beta X^N G_{MN} + S_{\text{bdy}} = -\sqrt{\lambda} , \tag{6.5}$$

where S_{bdy} is the boundary term needed to subtract the divergence according to a well understood and general prescription [15]. Note that the regularized area turns out to be negative. The strong coupling prediction for the Wilson loop expectation value is then

$$\langle W \rangle \simeq e^{\sqrt{\lambda}} . \tag{6.6}$$

This agrees with the strong coupling limit of the matrix model prediction (2.6). The factor of $\lambda^{-3/4}$ in (2.6) can be also argued to be consistent with the fact that there are 3 ghost zero modes on the disk (each zero mode carries a factor of $\lambda^{-1/4} = \sqrt{\alpha'}$) [7]. On the other hand, the overall numerical prefactor and the higher order corrections in (2.6) should be obtained by computing one-loop and higher loop corrections in the string σ-model. This is an open problem.

7 Supersymmetric Wilson loops as pseudo-holomorphic surfaces in AdS

In general it is difficult to find explicit solutions to the minimal area problem with given boundary conditions, but for supersymmetric Wilson loop it is actually possible to give a general characterization of the dual string worldsheets.

In the case of susy operators, one should require that the string solution, besides satisfying the boundary conditions (6.2), should also preserve exactly the same fraction of supersymmetries as the dual gauge theory operator. The supersymmetries preserved by a string solution are obtained by studying the κ-symmetry projection equation

$$\Gamma\epsilon_{AdS} = \epsilon_{AdS}\,, \qquad \Gamma = \frac{\sqrt{g}\epsilon^{\alpha\beta}\partial_\alpha X^M \partial_\beta X^N \Gamma_{MN}}{i\partial_\alpha X^M \partial^\alpha X^N G_{MN}}\,, \tag{7.1}$$

where Γ_M are 10d Dirac matrices, and ϵ_{AdS} is the $AdS_5 \times S^5$ Killing spinor, which reduces at the boundary $z = 0$ to the $\mathbb{R}^4$ conformal Killing spinor (1.3). This equation projects out some of the components of ϵ_{AdS} and tells us how many and which supersymmetries are preserved by the string solution.

Consider the case of the general Wilson loops on S^3 defined in section 4. First of all, since the operators only couple to 3 of the 6 scalars, we can restrict the string surface to lie on a $AdS_5 \times S^2$ subspace of $AdS_5 \times S^5$. On this space, it will be convenient to take the metric

$$ds^2 = \frac{1}{z^2}dx^\mu dx^\mu + z^2 dy^i dy^i\,, \qquad z^{-2} \equiv y^i y^i\,, \quad i = 1,2,3 \tag{7.2}$$

This is related to the more familiar product form of the $AdS_5 \times S^2$ metric (in Poincare patch) after separating out the radial part of the y^i coordinates. On this 7d space consider the 6d subspace $\mathcal{M}$ defined by

$$\mathcal{M} = \left\{(x^\mu, y^i) \in AdS_5 \times S^2 | x^\mu x^\mu + z^2 = 1\right\}\,. \tag{7.3}$$

It is easy to see that this is just $AdS_4 \times S^2$, with the boundary of AdS_4 being the S^3, defined by $x^\mu x^\mu = 1$, on which the Wilson loops live. It is then a natural guess that the strings dual to our Wilson loops on S^3 will lie in this 6d subspace. Indeed, by carefully studying the κ-symmetry projection (7.1) and matching the supersymmetries of strings and gauge theory operators, we find that the solutions do lie inside $\mathcal{M}$, and on this subspace they satisfy the first-order differential equation

$$J^M_N \partial_\alpha X^N = \sqrt{g}\epsilon_{\alpha\beta}\partial^\beta X^M\,, \tag{7.4}$$

where $X^M = (x^\mu, y^i)$ are the $AdS_5 \times S^2$ coordinates (constrained by (7.3)), and J^M_N is a 7×7 matrix which turns out to be an *almost complex structure* on $\mathcal{M} = AdS_4 \times S^2$, i.e.

$$J : T\mathcal{M} \to T\mathcal{M}\,, \qquad J^2 = -1 \;\text{ on } T\mathcal{M}\,. \tag{7.5}$$

The condition (7.4) is known as the statement that the string surfaces are pseudo-holomorphic with respect to the almost-complex structure J. It can be proved that solutions of (7.4) are automatically classical solutions of the $AdS_5 \times S^5$ σ-model. A similar result was obtained by Dymarsky et al [16] while studying the string theory duals of the Zarembo's supersymmetric Wilson loop.

The almost-complex structure we have found is not integrable, and it is closely related to the almost-complex structure of S^6. Recall that the S^6 is the only sphere besides S^2 to admit an almost-complex structure (in the case of S^2 the complex structure is of course integrable). Let us recall the construction of the S^2 complex structure in terms of the 3d vector product. If we think of S^2 as the subspace of $\mathbb{R}^3$ given by $x^i x^i = 1$, $i = 1, 2, 3$, the complex structure on S^2 can be defined in terms of the embedding coordinates as

$$J^i{}_j = \epsilon^i{}_{jk} x^k \,. \tag{7.6}$$

In vector notation, J acts on a vector $\vec{p} \in TS^2$ as $J(\vec{p}) = \vec{x} \times \vec{p}$, it clearly maps TS^2 to TS^2 and $J^2 = -1$ on TS^2. For S^6 one has a similar construction in terms of the vector product of the imaginary octonions. Embedding S^6 into $\mathbb{R}^7$ with coordinates x^a, $a = 1, \ldots, 7$, the almost-complex structure of S^6 is given by

$$J^a_{\ b} = \phi^a{}_{bc} x^c \,, \tag{7.7}$$

where $\phi^a{}_{bc}$ are the components of the G_2 associative 3-form (these are the structure constants for the imaginary octonions). The almost complex structure of $AdS_4 \times S^2$ related to our Wilson loops takes basically the same form as (7.7), modulo conformal z-factors. Its components may be written explicitly as

$$J^\mu_{\ \nu} = z^2 \sigma^i_{\mu\nu} y^i, \quad J^\mu_{\ i} = z^2 \sigma^i_{\mu\nu} x^\nu = -z^4 J^i{}_\mu, \quad J^i{}_j = -z^2 \epsilon_{ijk} y^k \,. \tag{7.8}$$

From the pseudo-holomorphicity equations (7.4) it also follows that the string solutions are calibrated by the 2-form with components $J_{MN} = G_{MP} J^P_{\ N}$, i.e. their area is simply given by

$$A(\Sigma) = \int_\Sigma J \,. \tag{7.9}$$

However note that, unlike a standard calibration, the 2-form J is not closed in our case (this is related to the fact that our Wilson loops have non-trivial expectation values).

7.1 The 1/8 BPS loops on S^2

To study the string theory duals to the 1/8 BPS loops on S^2 one simply restricts the general analysis done in the previous section to the $x^4 = 0$ subspace of $AdS_4 \times S^2$. This is just $AdS_3 \times S^2$, and the boundary of AdS_3 is the S^2 on which the operators of section 5 are defined. In this case the pseudo-holomorphicity conditions (7.4) may be written in the simpler form

$$\begin{aligned} & z^2 \partial_\alpha \left(\vec{x} \times \vec{y}\right) = \sqrt{g} \epsilon_{\alpha\beta} \partial^\beta \vec{x} \\ & \partial_\alpha \left(\vec{x} \cdot \vec{y}\right) = 0 \,. \end{aligned} \tag{7.10}$$

Explicit analytic solutions of these equations are currently known only for two special examples of loops on S^2: a latitude circle at arbitrary latitude angle θ_0 and a loop made of two arcs of longitude with an opening angle δ [3]. For example, the solution for a latitude circle is a relatively simple generalization of the 1/2 BPS solution described before, see eq. (6.4). The profile on $S^2 \subset S^5$ is now non-trivial, since the operator couples to 3 scalars. The coupling $\vec{\Theta} = \vec{x} \times \dot{\vec{x}}$ to the scalar fields describes in this case a "dual" latitude on $S^2 \subset S^5$ at angle $\pi/2 - \theta_0$, and the projection of the worldsheet on $S^2 \subset S^5$ is a spherical cap ending on this circle. The dominant solution is the one corresponding to the smaller cap, and its regularized area turns out to be

$$A_{\rm reg} = -\sqrt{\lambda}\sin\theta_0 = -\sqrt{\frac{4A_1A_2}{A^2}\lambda}\,, \tag{7.11}$$

where in the second equality we have used that $A_{1,2} = 2\pi(1 \mp \cos\theta_0)$ for the latitude loop. This is in precise agreement with the 2d YM/matrix model conjecture (5.3)-(5.4). Interestingly, the string surface wrapping the larger spherical cap on $S^2 \subset S^5$ (and with identical profile on AdS_3) is also a solution and it preserves the same supersymmetries. So it is a different saddle point which is dual to the same Wilson loop operator. This solution is unstable and its regularized area is equal and opposite to the one found above

$$A_{\rm reg}^{\rm unstable} = +\sqrt{\lambda}\sin\theta_0 = +\sqrt{\frac{4A_1A_2}{A^2}\lambda}\,. \tag{7.12}$$

This solution is clearly subdominant at strong coupling, but it is interesting that the presence of this additional unstable saddle point is also in precise agreement with the matrix model prediction. In fact, the asymptotic expansion of the Bessel function gives

$$\frac{2}{\sqrt{\lambda'}} I_1(\sqrt{\lambda'}) \stackrel{\lambda\gg 1}{\simeq} \sqrt{\frac{2}{\pi}}\frac{e^{\sqrt{\lambda'}}}{\lambda'^{3/4}}\left(1+\ldots\right) - i\sqrt{\frac{2}{\pi}}\frac{e^{-\sqrt{\lambda'}}}{\lambda'^{3/4}}\left(1+\ldots\right)\,, \tag{7.13}$$

where $\lambda' = \frac{4A_1A_2}{A^2}\lambda$ as before. It can be argued that the presence of two string solutions with equal and opposite regularized areas is in fact a general phenomenon which follows from the structure of the pseudo-holomorphicity equations (7.10) [4].

8 Invariance under area-preserving diffeomorphisms at strong coupling

Although we cannot find explicit analytic solutions of (7.10) for arbitrary loop, one can argue that the string action evaluated on pseudo-holomorphic solutions is invariant under deformations of the boundary loop $\mathcal{C}$ which preserve its area [4]. This is another non-trivial evidence for the conjectured relation to 2d YM.

Consider an arbitrary loop $\mathcal{C} \subset S^2$ and imagine that we have found the corresponding pseudo-holomorphic string solution Σ solving (7.10). Now consider an arbitrary deformation of the loop $\mathcal{C} \to \mathcal{C} + \delta\mathcal{C}$ such that the deformed loop still lies on S^2. Correspondingly, the string worldsheet will be deformed to a new solution $\Sigma + \delta\Sigma$ of the susy equations. We would like to determine how the string regularized area changes as a function of the loop variation δC. Since Σ and $\Sigma + \delta\Sigma$ are solutions, the difference of their on-shell action is a boundary term. To evaluate this boundary term, it is sufficient to solve the pseudo-holomorphic equations (7.10) perturbatively close to the boundary. This turn out to be possible for arbitrary loop, and the result is that [4]

$$S_{F1}[\Sigma + \delta\Sigma] - S_{F1}[\Sigma] = -c\,\delta A_1 \tag{8.1}$$

where δA_1 is just the variation of the area of the loop $\mathcal{C}$ at the boundary, and c is a constant. Therefore deformations of the loop which preserve areas leave the string worldsheet action invariant, and hence any two boundary loops with the same area correspond to string solutions with the same on-shell action (though the two explicit string solutions will be of course different in general). In particular, from the knowledge of the explicit solutions for circles of arbitrary radius (the latitudes described above), we can conclude that the regularized area of a given string solution (which may not be explicitly known) dual to a loop on S^2 of area A_1 will take the form

$$S^{\rm reg}_{F1} = \mp\sqrt{\frac{4A_1A_2}{A^2}\lambda}\,, \tag{8.2}$$

as predicted by the 2d YM/matrix model. It should be stressed that the result (8.1) crucially depends on the structure of the pseudo-holomorphic equations (7.10), and it certainly does not hold for general non-supersymmetric string solutions.

9 Correlators of two Wilson loops on S^2

As mentioned earlier, we can also study correlators of several Wilson loops. Consider for example two loops $\mathcal{C}_1$ and $\mathcal{C}_2$ on S^2. To compute the connected correlator $\langle W(\mathcal{C}_1)W(\mathcal{C}_2)\rangle_c$ from string theory, we are instructed to look for a solution to the supersymmetry equations (7.10) with topology of a cylinder whose boundaries are the two loops $\mathcal{C}_1$, $\mathcal{C}_2$.

For simplicity let us consider the case in which the two loops are two latitude circles on S^2 at angles θ_1 and θ_2. Making use of the circular symmetry of the problem, it is not difficult to see that actually there are no solutions of (7.10) with the topology of a smooth cylinder joining the two latitudes [4]. The only possibility is a "degenerate" cylinder made up of two disks, corresponding to two single latitude solutions, joined by a zero-area tube [5]. Physically, the zero-area tube corresponds to exchange of light supergravity modes between the two worldsheets. At leading order in the α' expansion, the connected correlator is approximated by the exponential of the classical area of this degenerate cylinder, which is just equal to the sum of the areas of the two disks, as given in eq. (7.11)

$$\langle W(\mathcal{C}_1)W(\mathcal{C}_2)\rangle_c \simeq \exp\left(\sqrt{\lambda}\sin\theta_1 + \sqrt{\lambda}\sin\theta_2\right)\,. \tag{9.1}$$

The exchange of supergravity modes between the two disks will contribute as a prefactor to the exponential and we do not compute it here. Using the invariance under area preserving diffeomorphisms discussed in the previous section, one can argue that the same conclusion (9.1) will hold for loops of arbitrary shape, with $\sin\theta_1 \to \sqrt{\frac{4A_1(A-A_1)}{A^2}}$.

We can now try to check whether the string theory prediction agrees with our 2d YM/matrix model conjecture. In the present case, the truncation of 2d YM to the zero-instanton sector leads to the following Gaussian Hermitian two-matrix model

$$\langle W_{R_1}(\mathcal{C}_1)W_{R_2}(\mathcal{C}_2)\rangle = \frac{1}{\mathcal{Z}}\int DX_1 DX_2\ \mathrm{tr}_{R_1}\, e^{X_1}\, \mathrm{tr}_{R_2}\, e^{X_2}\, e^{-\frac{A}{2g^2}\mathrm{tr}\left(\frac{1}{A_1}X_1^2+\frac{1}{A_{12}}(X_1-X_2)^2+\frac{1}{A_2}X_2^2\right)}\,. \tag{9.2}$$

Here A_1, A_{12} and A_2 are the three areas singled out by the two loops on S^2. This matrix model can be solved in the large N limit using e.g. the results of [17]. The end result for the connected correlator, in the case of $R_1 = R_2 = \text{fund.}$ takes the form

$$\begin{aligned}\langle W(\mathcal{C}_1)W(\mathcal{C}_2)\rangle_c &\overset{N\to\infty}{=} \sum_{n=1}^{\infty} n\rho^n I_n\left(\sqrt{\lambda_1'}\right) I_n\left(\sqrt{\lambda_2'}\right)\\ \lambda_1' = \frac{4A_1(A-A_1)}{A^2}\lambda\,, &\quad \lambda_2' = \frac{4A_2(A-A_2)}{A^2}\lambda\,, \quad \rho = \rho(A_1, A_2, A_{12})\,.\end{aligned} \tag{9.3}$$

Note that ρ is a function of the areas but not of the 't Hooft coupling. In the large λ limit, one can then extract the following strong coupling prediction

$$\langle W_{R_1}(\mathcal{C}_1)W_{R_2}(\mathcal{C}_2)\rangle_c \overset{\lambda\gg 1}{\simeq} \exp\left(\sqrt{\lambda_1'}+\sqrt{\lambda_2'}\right)\left[\frac{(\lambda_1'\lambda_2')^{-1/4}\rho}{2\pi(1-\rho)^2}+\dots\right]\,. \tag{9.4}$$

[5]It is very likely that a solution of the equations of motion with topology of a smooth cylinder with boundaries on the two latitudes does exist. However it will not be supersymmetric, i.e. it will not solve (7.10).

We see that the exponential saddle point agrees with the string picture of "disconnected" disks described above, see eq. (9.1). The prefactor to the exponential, as well as the higher order corrections, should be compared on the string theory side to the exchange of supergravity modes and to quantum fluctuations of the disks themselves. This is an interesting open problem.

10 Local operators and Wilson loops on S^2

We have seen that on S^2 we can put any number of Wilson loops of the form (5.2) so that the system is 1/8 BPS. It is actually possible to insert on the same sphere also an arbitrary number of local operators $O_J(x)$ such that the combined system still preserves some supersymmetry [5].

These local operators are defined as follows. Take the 3 scalars Φ^1, Φ^2, Φ^3 which couple to the Wilson loops, and any one of the 3 remaining scalars, say Φ^4. Then the local operators of interest are

$$O_J(x) = \mathrm{tr}\left(x^i\Phi^i + i\Phi^4\right)^J , \quad x^i \in S^2 . \tag{10.1}$$

These operators are chiral primaries, since they take the form $\mathrm{tr}(u\cdot\Phi)^J$ with $u^2 = 0$. However notice the slightly unusual feature that the vector u^I in scalar space is taken to be x-dependent. Being chiral primaries, each operator $O_J(x)$ preserves half of the Poincare supersymmetry, but operators inserted at different points preserve a different set of supersymmetries. However, it can be easily proved that the system of any number of operators (10.1) inserted at different points on the S^2 preserves at least 4 supersymmetries (4 combinations of Poincare and superconformal supercharges). Moreover, the combined system of any number of Wilson loops (5.2) and any number of local operators (10.1) still preserves 2 common supersymmetries, i.e. it is 1/16 BPS.

One of the two common supersymmetries is precisely the supercharge used by Pestun [14] in the localization calculation applied to the 1/8 BPS loops on S^2. Since $O_J(x)$ are Q-closed, localization to the 2d theory can be applied to mixed local-Wilson correlation functions of the form

$$\langle O_{J_1}(x_1)O_{J_2}(x_2)\cdots W_{R_1}(\mathcal{C}_1)W_{R_2}(\mathcal{C}_2)\cdots\rangle . \tag{10.2}$$

The localization equations found in [14] imply the following 4d-2d identification

$$O_J(x) \leftrightarrow \left(i *_{2d} F_{2d}(x)\right)^J , \tag{10.3}$$

i.e. our local operators are mapped in the 2d theory to insertions of powers of the YM field strength. Extending the conjecture stated earlier for Wilson loops, we then propose that the mixed correlation functions (10.2) can be computed exactly by the following correlator in the zero-instanton sector of 2d YM on S^2

$$\langle (i *_{2d} F_{2d}(x_1))^{J_1} \, (i *_{2d} F_{2d}(x_2))^{J_2} \cdots \mathrm{tr}_{R_1} P e^{\oint_{\mathcal{C}_1} A_{2d}} \, \mathrm{tr}_{R_2} P e^{\oint_{\mathcal{C}_2} A_{2d}} \cdots \rangle^{\text{0-inst}}_{YM_2} . \tag{10.4}$$

A simple non-trivial example is the case of the correlator of one local operator and one Wilson loop. From 2d YM, we expect that the correlator will only depend on the area of the loop. Moreover it should be almost independent on the insertion point of the local operator, i.e. it only depends on whether the operator is inserted "inside" or "outside" the loop (i.e. in the region containing the north or south pole). The truncation to zero-instantons in 2d YM produces in this case the following Gaussian Hermitian two-matrix model

$$\langle O_J(x) W_R(\mathcal{C})\rangle = \frac{1}{\mathcal{Z}} \int DXDY \,\mathrm{tr}\, Y^J \,\mathrm{tr}_R\, e^X e^{-\frac{A}{2g^2}\left(\frac{A_1}{A_2}\mathrm{tr}\, Y^2 - \frac{2i}{A_2}\mathrm{tr}\, XY\right)}, \qquad (10.5)$$

where we assumed that x sits in the region with area A_1 (if x sits in the region of area A_2 one simply exchanges $A_1 \leftrightarrow A_2$ and includes a factor $(-1)^J$ in the above equation). As a consistency check of this conjecture, notice that if we do not insert the local operator, then we can integrate out Y exactly and we recover the matrix model for a single loop (5.4).

It is easy to check that the above two-matrix model agrees with the leading order Feynman diagram computation in $\mathcal{N} = 4$ SYM. Furthermore, the matrix model can be solved exactly, see [5] for details, and in particular we can extract the strong coupling behavior and compare to string theory. In the planar limit and at large λ, we obtain from (10.5) the following prediction in the case of Wilson loop in the fundamental (the normalization by $\langle W(\mathcal{C})\rangle$ is for convenience)

$$\frac{\langle O_J(x) W(\mathcal{C})\rangle}{\langle W(\mathcal{C})\rangle} \overset{\lambda\to\infty}{\sim} \sqrt{J\lambda}\left(\frac{A_2}{A}\right)^{\frac{J+1}{2}}\left(\frac{A_1}{A}\right)^{\frac{-J+1}{2}}\left(1+\mathcal{O}(\frac{1}{\sqrt{\lambda}})\right). \qquad (10.6)$$

This result can be successfully reproduced by a string theory calculation, including all numerical factors. In string theory, the local operators are dual to the J-th KK-mode on S^5 of a certain supergravity scalar field which is a linear combination of fluctuations of the metric and the Ramond-Ramond 4-form potential [18][19]. The correlator at strong coupling is computed by evaluating the amplitude for the process in which the supergravity mode dual to $O_J(x)$ is emitted from the insertion point x at the boundary and then absorbed at a point on the string worldsheet dual to the Wilson loop. Schematically, the string theory answer takes the form

$$\frac{\langle O_J(x) W(\mathcal{C})\rangle}{\langle W(\mathcal{C})\rangle} = \sqrt{\lambda}\int_\Sigma d^2\sigma V_J(X(\tau,\sigma);x) G_J(X(\tau,\sigma);x) \qquad (10.7)$$

where $X(\tau,\sigma)$ denotes the string solution, V_J is a "vertex operator" which is obtained by computing how the supergravity mode dual to O_J couples to the worldsheet, and finally G_J is the standard bulk-to-boudnary propagator, describing the propagation of the supergravity mode from the string worldsheet to the insertion point at the boundary. The integral (10.7) can be computed in the case of the explicitly known solutions (the "latitude" and "two-longitudes" loops mentioned in section 7.1), and the result precisely agrees with the prediction (10.6). In particular, it is quite non-trivial from the string theory calculation that the correlator turns out to be (almost) independent on the insertion point x of the local operator, as implied by the 2d YM conjecture.

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