Grand Unification

Lecture notes

January 28, 2007

1 Introduction

1.1 Unanswered questions within the SM

In spite of all the successes of the Standard Model, it is unlikely to be the final answer, it leaves many unanswered questions:

- Why three forces? Why simple groups?
- Why does one family consist of the states $[Q, u^c, d^c; L, e^c]$ transforming as [(3, 2, 1/6), (3, 1,?2/3), (3, 1, 1/3); (1, 2,1), (1, 1, 2)], where $Q = (u, d)$, and $L = (\nu, e)$ are $SU(2)_L$ doublets, and u^c, d^c, e^c are charge conjugate $SU(2)_L$ singlet fields with the $U(1)_Y$ quantum numbers given? [We use the convention that electric charge $Q_{EM} = T^3_L + Y$ and all fields are left-handed.]
- Why the local gauge interactions $SU(3)_C?SU(2)_L?U(1)_Y$?
- Why charged is quantize in units of 1/3? No fractional charged hadrons?
- Neutrino masses?
- Anomaly cancellation? The way that the anomalies cancel within a single generation looks really miraculous: U(1)$_Y$-SU(3)^2:

$$\left(2Q_Y^{Q_l} - Q_Y^{u_l} - Q_Y^{d_l}\right) = 0\,, \tag{1}$$

  where the subscript $l = 1, 2, 3$ stands for the three generations.
  U(1)$_Y$-SU(2)^2:

$$\left(3Q_Y^{Q_l} - Q_Y^{L_l}\right) = 0\,. \tag{2}$$

  U(1)^3_Y:

$$6\left(Q_Y^{Q_l}\right)^3 - 3\left(Q_Y^{u_l}\right)^3 - 3\left(Q_Y^{d_l}\right)^3 + 2\left(Q_Y^{L_l}\right)^3 - \left(Q_Y^{l_l}\right)^3 = 0\,. \tag{3}$$

U(1)$_Y$-G^2 (gravity related anomaly):

$$\left(6Q_Y^{Q_l} - 3Q_Y^{u_l} - 3Q_Y^{d_l} + 2Q_Y^{L_l} - Q_Y^{l_l} - Q_Y^{N_l}\right) = 0\,. \tag{4}$$

and also the flavor puzzle, the hierarchy problem, baryogenesis, dark matter, the strong CP problem... The itemized question are the problems which grand unified theories (GUTS) hope to address (not necessarily the 1st priority ...).

If all of our forces unifies how come their strength is so different:

$$\begin{aligned} \alpha_Y(MZ) &= 0.017, \\ sin^2\theta_W(MZ) &= 0.2312 \quad or \quad \alpha_2(MZ) = 0.034, \\ \alpha_3(MZ) &= 0.117\,. \end{aligned} \tag{5}$$

How come they are unified (we shall use either $U(1)_Y$ or $U(1)_1$ for the hypercharge abelian interaction)??

The answer is due to the fact that the gauge couplings (as other ones) change as a function of scale this is described via the renormalization group equations (RGE):

$$\frac{d}{dt}\alpha_i^{-1} = -b_i/2\pi \qquad (i = 1, 2, 3)\,,, \tag{6}$$

where $t = \ln \mu/\mu_0$. We then extrapolate these couplings to higher energy scales μ via the standard one-loop renormalization group equations (RGE's) of the form

$$\alpha_i^{-1}(\mu) \;=\; \alpha_i^{-1}(M_Z) \;-\; \frac{b_i}{4\pi}\,\ln\,\frac{\mu^2}{M_Z^2}\;. \tag{7}$$

Note that the one-loop beta-function coefficients b_i that govern this logarithmic running depend on the matter content of the theory. Specifically, one finds that these coefficients take the following values, within the Standard Mode

$$(b_1, b_2, b_3) \simeq\; (41/10, -19/6, -7) \tag{8}$$

Using the beta-function coefficients b_i of the Standard Model and extrapolating the low-energy couplings upwards according to Eq. (7), one then finds that the three gauge couplings seems to converge and almost meet at around 10^{14-15} GeV, a remarkable fact. However a more precise examination shows that they fail to meet at any scale. This is illustrated in Fig. 1.

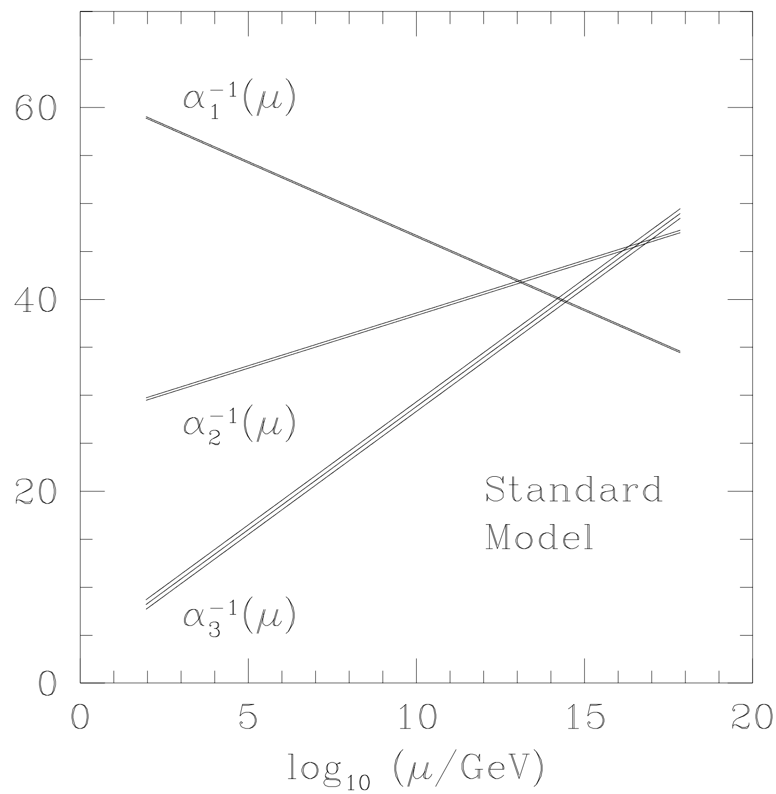

Figure 1: One-loop evolution of the gauge couplings within the (non-supersymmetric) Standard Model. Here $\alpha_1 \equiv (5/3)\alpha_Y$ the hypercharge hyperfine constant and α_2 is the weak one in the conventional normalization. The relative width of each line reflects current experimental uncertainties.

2 GUT

The problems mentioned above can be partly solved by assuming the symmetry groups $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ are part of a larger group G, i.e.

$$G \supset SU(3)_C \otimes SU(2)_L \otimes U(1)_Y. \tag{9}$$

The SM is rank 4 so the smallest possible simple group G is the SU(5) group[1].

So the minimal extension of the SM towards a GUT is based on the $SU(5)$ group. Throughout this paper we will mainly consider this minimal extension, for simplicity.

The group G has a single coupling constant for all interactions and the observed differences in the couplings at low energy are caused by radiative corrections. As discussed before, the strong coupling constant decreases with increasing energy, while the electromagnetic one increases with energy, so that at some high energy they will become equal. Since the changes with energy are only logarithmic, the unification scale is high, namely of the order of $10^{15} - 10^{16}$ GeV, depending on the assumed particle content in the loop diagrams.

Since GUT "only" aims towards unify the gauge couplings then we can look for a rep' for a single SM generation. One generation contains 15 degrees of freedom. The possible representations are $5 + \bar{5} + 10 + \bar{1}0 + 15 + \bar{1}5$ where higher representations require extra degrees of freedom.

It is remarkable that within the $SU(5)$ group the 15 particles and antiparticles of a single generation can be fit into the $\overline{5}$-plet[2] and 10-plet:

$$\overline{5} = \begin{pmatrix} d_g^c \\ d_r^c \\ d_b^c \\ e^- \\ -\nu_e \end{pmatrix} \quad 10 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & +u_b^c & -u_r^c & -u_g & -d_g \\ -u_b^c & 0 & +u_g^c & -u_r & -d_r \\ +u_r^c & -u_g^c & 0 & -u_b & -d_b \\ +u_g & +u_r & +u_b & 0 & -e^+ \\ +d_g & +d_r & +d_b & +e^+ & 0 \end{pmatrix}_L \tag{10}$$

The superscript c indicates the charge conjugated particle, i.e. the antiparticle and all particles are chosen to be left-handed, since a left-handed antiparticle transforms like a right-handed particle. Thus the superscript c implies a right-handed singlet with weak isospin equal zero.

With this multiplet structure the sum of the quantum numbers Q, T_3 and Y is zero within one multiplet, as required, since the corresponding operators are represented by traceless matrices.

$SU(5)$ rotations can be represented by 5×5 matrices. Local gauge invariance requires the introduction of $5^2 - 1 = 24$ gauge fields, which cause the interactions between the

[1] G cannot be the direct product of the $SU(3)$, $SU(2)$ and $U(1)$ groups, since this would not represent a new unified force with a single coupling constant, but still require three independent coupling constants.

[2] The bar indicates the complementary representation of the fundamental representation.

matter fields. The gauge fields transform under the adjoint representation of the $SU(5)$ group, which can be written in matrix form as

$$
24 = \left(\begin{array}{ccc|cc}
G_{11} - \frac{2B}{\sqrt{30}} & G_{12} & G_{13} & X_1^C & Y_1^C \\
G_{21} & G_{22} - \frac{2B}{\sqrt{30}} & G_{23} & X_2^C & Y_2^C \\
G_{31} & G_{32} & G_{33} - \frac{2B}{\sqrt{30}} & X_3^C & Y_3^C \\
\hline
X_1 & X_2 & X_3 & \frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} & W^+ \\
Y_1 & Y_2 & Y_3 & W^- & -\frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}}
\end{array} \right) \tag{11}
$$

The G's represent the gluon fields while the W's and B's are the gauge fields of the $SU(2)$ symmetry groups.

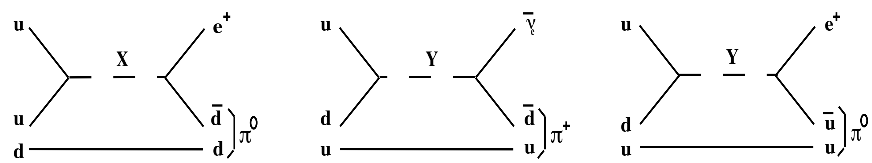

Figure 2: GUT proton decays through the exchange of X and Y gauge bosons.

3 $SU(5)$ predictions

The presence of the gauge fields X, Y yields several dramatic phenomenological consequences. The X and Y's are new gauge bosons, which represent interactions, in which quarks are transformed into leptons and vice-versa, as should be apparent if one operates with this matrix on the $\overline{5}$-plet.

Consequently, the X (Y) bosons, which couple to the electron (neutrino) and d-quark must have electric charge 4/3 (1/3).

Such fields with non-standard electric charge assignment are denoted as exotic fields.

3.1 Proton decay

On top of carrying a non-standard electric charges, the X and Y gauge bosons can introduce transitions between quarks and leptons!

Thus they violate lepton and baryon number.

Note though, that the difference between lepton and baryon number $B-L$ is conserved in these transitions.

This can lead to the following proton and neutron decays (see fig. 2):

$$
\begin{array}{cc}
p \to e^+\pi^0 & n \to e^+\pi^- \\
p \to e^+\rho^0 & n \to e^+\rho^- \\
p \to e^+\omega^0 & n \to \nu\omega^0 \\
p \to e^+\eta & n \to \overline{\nu}\pi^0 \\
p \to \overline{\nu}\pi^+ & n \to \overline{\nu}_\mu K^0 \\
p \to \overline{\nu}\rho^+ & \\
p \to \overline{\nu}_\mu K^+ &
\end{array}
\tag{12}
$$

The decays with kaons in the final state are allowed through flavour mixing, i.e. the interaction eigenstates are not necessarily the mass eigenstates.

For the lifetime of the nucleon one writes in analogy to muon decay:

$$\tau_p \approx \frac{M_X^4}{\alpha_5^2 m_p^5} \tag{13}$$

($t_{\mathrm{Pl}} = 1/M_{\mathrm{Pl}} \sim 5 \times 10^{-44} s \sim 10^{-51} yr; \tau_p/t_{\mathrm{Pl}} = \frac{M_X^4 M_{\mathrm{Pl}}}{\alpha_5^2 m_p^5}$) The proton mass m_p to the fifth power originates from the phase space in case the final states are much lighter than the proton, which is the case for the dominant decay mode: $p \to e^+\pi_0$. After this prediction of an unstable proton in grand unified theories, a great deal of activity developed and the lower limit on the proton life time increased to

$$\tau_p \geq 10^{33} \text{ yr} \tag{14}$$

for the dominant decay mode $p \to e^+\pi^0$. From equation 13 this implies (for $\alpha_5 = 1/24$, the result of the fit)

$$M_X \geq 10^{15} \text{ GeV}. \tag{15}$$

From the extrapolation of the couplings in the $SU(5)$ model to high energies one expects the unification scale to be reached well below 10^{15} GeV, so the proton lifetime measurements exclude the minimal $SU(5)$ model as a viable GUT. As will be discussed later, the supersymmetric extension of the $SU(5)$ model has the unification point well above 10^{15} GeV.

3.2 Baryon Asymmetry

The heavy gauge bosons responsible for the unified force cannot be produced with conventional accelerators, but energies above 10^{15} were easily accessible during the birth of our universe. This could have led to an excess of matter over antimatter right at the beginning, since the X and Y bosons can decay into pure matter, e.g. $X \to uu$, which is allowed because the charge of the X boson is 4/3. As pointed out by Sakharov such an excess is possible if both C and CP are violated, if the baryon number B is violated, and if the process goes through a phase of non-equilibrium. All three conditions are possible within the $SU(5)$ model. The non-equilibrium phase happens if the hot universe cools down and arrives at a temperature, too low to generate X and Y bosons anymore, so only the decays are possible. Since the CP violation is expected to be small, the excess of matter over antimatter will be small, so most of the matter annihilated with antimatter into enormous number of photons. This would explain why the number of photons over baryons is so large:

$$\frac{N_\gamma}{N_b} \approx 10^{10} \tag{16}$$

However, later it was realized that the electroweak phase transition may wash out any (B+L) excess generated by GUT's. One then has to explain the observed baryon asymmetry by the electroweak baryogenesis, which is actively studied.

In practice due to the monopole problem this is probably not a viable scenario.

3.3 Charge Quantization

From the fact that quarks and leptons are assigned to the same multiplet the charges must be related, since the trace of any generator has to be zero. For example, the charge operator Q on the fundamental representation yields:

$$TrQ = Tr(q_{\overline{d}}, q_{\overline{d}}, q_{\overline{d}}, q_e, q_{\nu_e}) = 0 \tag{17}$$

or in other words, in $SU(5)$ the electric charge of the d-quark has to be 1/3 of the charge of an electron! Similarly, one finds the charge of the u-quark is 2/3 of the positron charge, so the total charge of the proton (=uud) has to be exactly opposite to the charge of an electron.

3.4 Prediction of $\sin^2\theta_W$

If the $SU(2)$ and $U(1)$ groups have equal coupling constants, the electroweak mixing angle can be calculated easily, since it is given by the ratio $g'^{\,2}/(g^2 + g'^{\,2})$ (see eq. **??**), which would be 1/2 for equal coupling constants. However, the argument is slightly more subtle, since for unitary transformations the rotation matrices have to be normalized such that

$$Tr\ (F_k F_l) = \delta_{kl}. \tag{18}$$

This normalization is not critical in case one has independent coupling constants for the subgroups, since a "wrong" normalization for a rotation matrix can always be corrected by a redefinition of the corresponding coupling constants. This freedom is lost, if one has a single coupling constant, so one has to be careful about the relative normalization. It turns out, that the Gell-Mann and Pauli rotation matrices of the $SU(3)$ and $SU(2)$ groups have the correct normalization, but the normalization of the weak hypercharge operator needs to be changed. Defining $Y = CT_0$ and substituting this into the SM relation between electric charge and isospin/hypercharge yields:

$$Q = T_3 + CT_0 \tag{19}$$

Requiring the same normalization for T_3 and T_0 implies from equation 18:

$$Tr\ Q^2 = (1 + C^2)\ Tr\ T_3^2\,. \tag{20}$$

Inserting the numbers from the $\overline{5}$-plet of $SU(5)$ yields:

$$1 + C^2 = \frac{Tr\ Q^2}{Tr\ T_3^2} = \frac{3 \cdot 1/9 + 1}{2 \cdot 1/4} = \frac{8}{3}. \tag{21}$$

Replacing in the covariant derivative Y_W with CT_0 implies $g'CT_0 \equiv g_5 T_0$ or:

$$g_5 = Cg', \tag{22}$$

where $C^2 = 5/3$ from eq. 21. With this normalization the electroweak mixing angle after unification becomes:

$$\sin^2\theta_W = \frac{g'^2}{(g^2 + g'^{\,2})} = \frac{g_5^2/C^2}{(g_5^2 + g_5^2/C^2)} = \frac{1}{1 + C^2} = \frac{3}{8}. \tag{23}$$

The manifest disagreement with the experimental value of 0.23 at low energies brought the $SU(5)$ model originally into discredit, until it was noticed that the running of the couplings between the unification scale and low energies could reduce the value of $\sin^2\theta_W$ considerably. As we will show in the last chapter, with the very precise measurement of $\sin^2\theta_W$ at LEP, unification of the three coupling constants within the $SU(5)$ model is excluded, and just as in the case of the proton life time, supersymmetry comes to the rescue and unification is perfectly possible within the supersymmetric extension of $SU(5)$.

Note that the prediction of $\sin^2\theta_W = 3/8$ is not specific to the $SU(5)$ model, but is true for any group with $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ as subgroups, implying that Q, T_3 and Y are generators with traces equal zero and thus leading to the predictions given above.

4 Spontaneous Symmetry Breaking in $SU(5)$

The $SU(5)$ symmetry is certainly broken, since the new force corresponding to the exchange of the X and Y bosons would lead to very rapid proton decay, if these new gauge bosons were massless. As mentioned above, from the limit on the proton life time these $SU(5)$ gauge boson have to be very heavy, i.e. masses above 10^{15} GeV. The generation of masses can be obtained again in a gauge invariant way via the Higgs mechanism. The Higgs field is chosen in the adjoint representation $\underline{24}$ and the minimum $< \Phi_{24} >$ can be chosen in the following way:

$$< \Phi_{24} >= v_{24} \left(\begin{array}{ccc|cc} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ \hline & & & -\frac{3}{2} & \\ & & & & -\frac{3}{2} \end{array} \right) \tag{24}$$

The 12 X,Y gauge bosons of the $SU(5)$ group require a mass:

$$M_X^2 = M_Y^2 = \frac{25}{8} g_5^2 v_{24}^2 \tag{25}$$

after 'eating' 12 of the 24 scalar fields in the adjoint representation, thus providing the longitudinal degrees of freedom. The field Φ_{24} is invariant under the rotations of the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ group, so this symmetry is not broken and the corresponding gauge bosons, including the W and Z bosons, remain massless. after the first stage of $SU(5)$ symmetry breaking.

The usual breakdown of the electroweak symmetry to $SU(3)_C \otimes U(1)_{em}$ is achieved by a 5-plet Φ_5 of Higgs fields, for which the minimum of the effective potential can be chosen at:

$$< \Phi_5 >= v_5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{26}$$

The fourth and fifth component of Φ_5 correspond to the $SU(2)$ doublet (Φ^+, Φ^0) of the SM. Since the total charge in a representation has to be zero again, the first triplet of complex fields in Φ_5, which transforms as $(3,1)_{-2/3}$ and $(3^*, 1)_{2/3}$, must have charge $|1/3|$. Since they couple to all fermions with mass, they can induce proton decay:

$$u + d \rightarrow H^{1/3} \rightarrow \begin{array}{c} e^+ + \overline{u} \\ \overline{\nu}_e + \overline{d} \end{array} \tag{27}$$

Such decays can be suppressed only by sufficiently high masses of the coloured Higgs triplet. These can obtain high masses through interaction terms between Φ_5 and Φ_{24}.

Note that from eq. (25) $< \Phi_{24} >$ has to be of the order of M_X, while Φ_5 has to be of the order of M_W, since

$$M_W^2 = \frac{1}{2} (g_5 v_5)^2 \tag{28}$$

and

$$M_Z = \frac{M_W}{\cos \theta_W} \tag{29}$$

or more precisely $v_5 = 1/\sqrt{G_F} = 174$ GeV. The minimum of the Higgs potential involves both Φ_5 and Φ_{24}. Despite this mixing, the ratio $v_5/v_{24} \approx 10^{-13}$ has to be preserved (hierarchy problem). Radiative corrections spoil usually such a fine-tuning, so $SU(5)$ is in trouble. This problem is sometimes denoted as doublet triplet mixing.

5 Relations between Quark and Lepton Masses

The Higgs 5-plet Φ_5 can be used to generate fermion masses. Since the $\overline{5}$-plet of the matter fields contains both leptons and down-type quarks, their masses are related, while the up-type quark masses are free parameters. At the GUT scale one expects:

$$m_d = m_e \tag{30}$$

$$m_s = m_\mu \tag{31}$$

$$m_b = m_\tau \tag{32}$$

which gives $m_s/m_d = m_\mu/m_e$. The relation is a renormalization group invariant, and is thus satisfied at any scale. This relation is in serious disagreement with the data, namely $m_s/m_d \sim 20$ and $m_\mu/m_e \sim 200$. The b-quark mass can be correctly predicted from the τ-mass after including radiative corrections. Since the corrections from graphs involving the strong coupling constant α_s are dominant, one expects in first order

$$\frac{m_b}{m_\tau} = \mathcal{O}\left(\frac{\alpha_s(m_b)}{\alpha_s(M_X)}\right) = \mathcal{O}(3)\,, \tag{33}$$

which is more reasonable.