Gauge Theories from Exceptional Collections

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Guiding Question

What is the low energy gauge theory description of a stack of D-branes probing a Calabi-Yau singularity?

Important question:

1. Strongly coupled gauge theories from geometry
2. String theory in curved backgrounds from gauge theory
3. D-branes

Strongly Coupled Gauge Theories

Placing D3- and D5-branes at the tip of the conifold $\sum_{i=1}^{4} z_i^2 = 0$ led to the Klebanov-Strassler solution and a geometric notion of

- chiral symmetry breaking as deformation
- confining flux tubes as fundamental strings
- renormalization group flow as extra RR flux

Faith that more is waiting to be discovered. The solutions I discuss today are all cousins of Klebanov-Strassler.

Flux Vacua

Thinking of the singularity as a local feature of a compact manifold, and replacing the D-branes by fluxes leads to warped flux compactifications of type IIB supergravity with all complex structure moduli stabilized.

Stabilizing the Kähler moduli as well (instantons, α' corrections, etc.), one has a "realistic" string theory vacuum that might model real world physics.

In this sense, Klebanov-Strassler (and by analogy its cousins) underlies the string theory landscape.

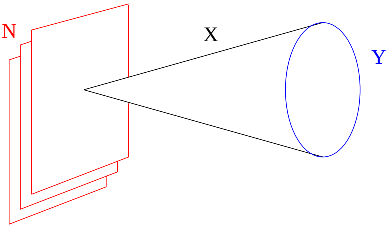
Outline

1. A bestiary of singular Calabi-Yaus
2. An argument for Exceptional Collections
3. An application – the $L^{p,q,r}$

C. H., R. Karp, "Exceptional collections and D-branes probing toric singularities," hep-th/0507175
C. H.,"Seiberg duality is an exceptional mutation," hep-th/0405118

A Bestiary of Singular Calabi-Yaus

Place a stack of D-branes at the tip of a six dimensional Calabi-Yau cone X in type IIB string theory



- D3-brane $= \mathbb{R}^{3,1} +$ point
- D5-brane $= \mathbb{R}^{3,1} +$ complex curve
- D7-brane $= \mathbb{R}^{3,1} +$ complex surface

The Bestiary Continued

Three sets of examples where the gauge theory can be derived

- X is an orbifold, i.e. $\mathbb{C}^3/\Gamma$ where Γ is a discrete subgroup
- X is toric – 3 U(1) isometries (can be related to Abelian orbifolds)
- X is a $\mathbb{C}$-cone over a del Pezzo

First Came the Orbifolds

- $X = \mathbb{C}^3$ leads to $\mathcal{N} = 4$ $U(N)$ super Yang-Mills theory
- Use representation theory to understand $\mathbb{C}^3/\Gamma$ where $\Gamma \subset SU(3)$ is a discrete group
- Douglas and Moore, Sardo-Infirri, ...

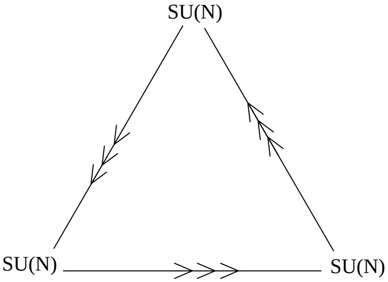


Figure: The quiver for the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold.

Toric Methods

- X has 3 $U(1)$ isometries

$$X = \frac{\mathbb{C}^q - F_\Delta}{(\mathbb{C}^*)^{q-3}}$$

- Partial resolution of $\mathbb{C}^3/\mathbb{Z}_n \times \mathbb{Z}_n$ looks like a Higgsing procedure on the gauge theory side
- Algorithmic dimer methods: For toric manifolds, the quiver can be drawn on a torus whose dual graph is a dimer model!
- Morrison, Plesser, Greene, Hanany, ...

Exceptional Collections

Finding a good basis of D-branes

- any case where the singularity can be partially resolved by blowing up a possibly singular compact $\mathbb{C}$-surface
- originally developed for the del Pezzos
- can handle many toric cases as well, e.g. the L^{pqr} spaces of Cvetic, Lu, Page, and Pope

Inter-relationships

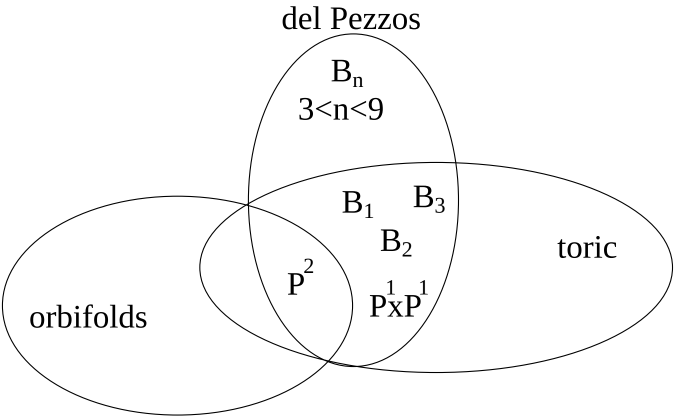


Figure: Different non-compact Calabi-Yau singularities

A Suggestive Theorem

The set of X is truly vast as demonstrated by a theorem due to Tian and Yau.

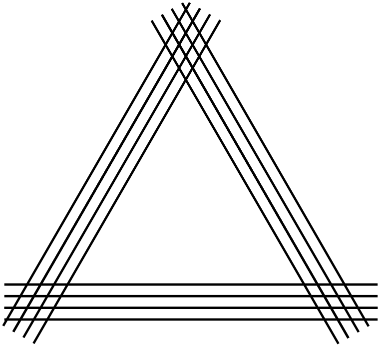
Let $f(x_1, x_2, x_3, x_4)$ be a polynomial in four complex variables x_i such that under the scaling $x_i \to \lambda^{w_i} x_i$, f transforms homogenously: $f \to \lambda^d f$. If $\sum_i w_i - d > 0$, then the manifold f=0 is Calabi-Yau. (The associated variety in weighted projective space must be Kähler-Einstein.)

Examples

- $f = x + y + z + w$ gives $\mathbb{C}^3$
- $f = x^2 + y^2 + z^2 + w^2$ yields the conifold

Gauge Theories from D-branes

- D-branes placed at a singularity break up into pieces – the fractional branes.
- For each stack of n fractional branes, we get a $U(n)$ gauge theory
- Where one stack overlaps another (massless open strings), we get bifundamental matter fields



Between String Field Theory and a Cartoon

What is the minimum information about the D-brane we need to retain to derive this gauge theory rigorously?

1. 1st-pass: D-branes as vector bundles?
2. 2nd-pass: D-branes as sheaves?
3. 3rd-pass: D-branes as complexes of sheaves?

1st pass: D-branes as vector bundles

A D-brane is some submanifold and on that submanifold we have a gauge field strength

BUT,

- we want a definition on X, not for $M \subset X$
- we want a definition for bound states of branes of different dimension

Between String Field Theory and a Cartoon

What is the minimum information about the D-brane we need to retain to derive this gauge theory rigorously?

1. 1st-pass: D-branes as vector bundles? NO
2. 2nd-pass: D-branes as sheaves?
3. 3rd-pass: D-branes as complexes of sheaves?

2nd pass: D-branes as sheaves

Sheaves provide a natural way of combining vector bundles of different rank (read D-branes of different dimension).

$$\mathcal{O}, \mathcal{O}_D, \mathcal{O}_{pt}$$

$$0 \to \mathcal{O}(-D) \to \mathcal{O} \to \mathcal{O}_D \to 0$$

BUT, we need anti-branes in addition to branes

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3rd pass: D-branes as complexes of sheaves

For E a sheaf,

$$\delta E \equiv \cdots \to 0 \to 0 \to E \to 0 \to \cdots$$

$$\delta E[1] \equiv \cdots \to 0 \to E \to 0 \to 0 \to \cdots$$

For a brane A, the antibrane is $A[1]$.
After carefully worrying about which complexes are equivalent as D-branes, we arrive at $D^b(X)$.

Categorical Description

- A category is a set of objects and maps between those objects
- These fractional branes $(A_1, A_2, \ldots A_m)$ can be defined as objects in the derived category of coherent sheaves on X, $D^b(X)$.
- The bifundamental matter fields are the maps between these objects (so called Ext maps or generalized sheaf cohomology).

The Open Strings

Bifundamental matter:

1. First pass: thinking massless $\sim$ topological, one might guess that for two holomorphic vector bundles V and W

$$H^q(X, V^* \otimes W) \quad q = 0, 1, 2, 3$$

2. Second pass: While E^* of a sheaf is not so well defined, one has instead $\mathrm{Ext}^q_X(E, F)$ where for vector bundles V and W corresponding to sheaves E and F

$$\mathrm{Ext}^q_X(E, F) = H^q(X, V^* \otimes W)$$

3. Third pass: For objects in $D^b(X)$

$$\mathrm{Ext}^k_X(E, F) = \mathrm{Hom}^{k-p+q}_{D^b(X)}(\delta E[p], \delta F[q])$$

The Fractional Branes

Moral definition:

- We should be able to reassemble any collection of D3-, D5-, and D7-branes from our complete set of fractional branes.
- The branes should be mutually supersymmetric.
- There should be a gauge field living on each fractional brane.

Definition: A set of fractional branes $\mathcal{A} = (A_1, A_2, \ldots, A_n)$ on X satisfy the following properties

- For any $A \in \mathcal{A}$, $\mathrm{Hom}^0_{D^b(X)}(A, A) \neq 0$.
- For any $A, B \in \mathcal{A}$, $A \neq B$, $\mathrm{Hom}^q_{D^b(X)}(A, B) = 0$ for $q = 0$ and $q = 3$.
- The set $\mathcal{A}$ generates $D^b(X)$.

Finding the Fractional Branes

Exceptional collections provide a way of finding a set of fractional branes.

Claim: Assume we can partially resolve X while keeping X Calabi-Yau. In the case where the resolution involves blowing up a complex surface, if we can find a (strong) exceptional collection of coherent sheaves on that surface, the collection lifts to a set of fractional branes on X.

Example: For the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold, the singularity can be completely eliminated by blowing up a $\mathbb{P}^2$. There is a strong exceptional collection on $\mathbb{P}^2$ of the form $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2)$ which lifts to the quiver gauge theory of before.

The Definition of an Exceptional Collection

Recall that for line bundles

$$H^q(X, \mathcal{O}(-E+F)) = \mathrm{Ext}^q_X(\mathcal{O}(E), \mathcal{O}(F))$$

Def: An exceptional sheaf E has

- $\mathrm{Ext}^0(E, E) = \mathbb{C}$
- $\mathrm{Ext}^q(E, E) = 0$ for $q > 0$

Def: For an exceptional collection $\mathcal{E} = (E_1, E_2, \ldots, E_n)$

- E_i is exceptional $\forall i$
- $\mathrm{Ext}^q(E_i, E_j) = 0$ for $i > j$

For a strong exceptional collection, additionally

- $\mathrm{Ext}^q(E_i, E_j) = 0$ for $i < j$ and $q > 0$

The L^{abc} manifolds

1. A toric Calabi-Yau 3-fold singularity can be described by a set of n coplanar vectors $V_i \in \mathbb{Z}^3$.
2. The L^{abc} manifolds correspond to the case where $n = 4$ such that

$$aV_1 + bV_3 = cV_2 + dV_4$$

   where $a + b = c + d$ and the a, b, c and d are positive integers.
3. Explicit metrics on these manifolds were recently discovered by Cvetič, Lu, Page, and Pope.

Some Simple Examples

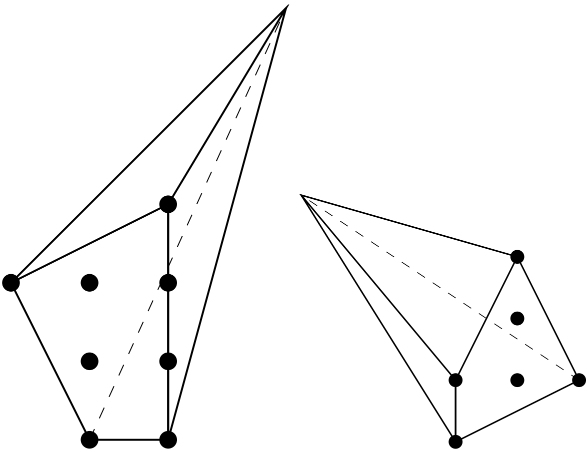


Figure: The cones for L^{263} and L^{152}.

Continuing with L^{152}.

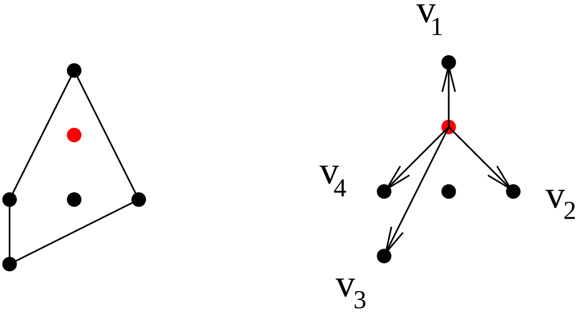
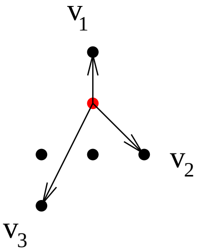


Figure: We blow up the surface V corresponding to the red dot. The surface is described by the vectors v_i on the left. Note that $v_4 = v_1 + v_3$.

An exceptional collection for L^{152}.

Removing the vector v_4 from the fan corresponds to blowing down a S^2 on our surface, resulting in $\mathbb{P}(1,1,3)$:



where $v_2 + v_3 + 3v_1 = 0$.
An exceptional collection on $\mathbb{P}(1,1,3)$ is

$$\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3), \mathcal{O}(4)$$

A strong exceptional collection on V is then

$$\mathcal{O}, \mathcal{O}(E), \mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(3), \mathcal{O}(4)$$

where E is the divisor corresponding to the S^2.

Calculating the quiver for L^{152}

We compute

$$S_{ij} = \dim \mathrm{Hom}(E_i, E_j) = \begin{pmatrix} 1 & 1 & 2 & 3 & 5 & 7 \\ 0 & 1 & 1 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The inverse matrix encodes the maps between the A_i in the quiver:

$$S^{-1} = \begin{pmatrix} 1 & -1 & -1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -2 & 1 & -1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The L^{152} quiver

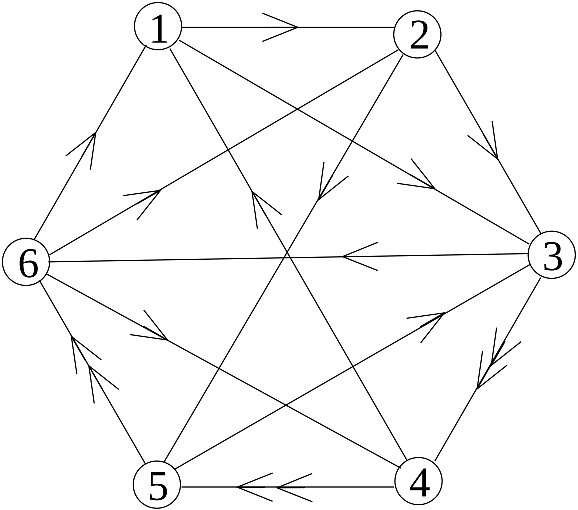


Figure: The quiver for the L^{152} singularity.

Conclusion

Hope to have left you with three ideas:

1. Figuring out the gauge theory description of D-branes probing a singularity is an important task.
2. Exceptional collections are, at the moment, the best way of deriving these gauge theories.
3. A general appreciation for the kind of arguments that lie behind the collections.