Matrices and Matrix Multiplication

We start with an $m \times n$ matrix A, that is, an array of numbers with m rows and n columns:

$$A = (A_{ij}) = \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1n} \\ A_{21} & A_{22} & \ldots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \ldots & A_{mn} \end{pmatrix}.$$

A_{ij} is the *element* of the matrix A in the ith row and jth column, *e.g.,* if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad \text{then} \quad A_{23} = 6.$$

So far, a matrix is just an array of numbers like this, it doesn't *mean* anything. As we see more and more physics, we will attribute more and more meaning to matrices and their manipulations. Your first problem in PS 1 is the first step in this process.

We multiply matrices by simply following what appears to be an arbitrary rule. (You will be seeing many "new ways to multiply," in your math courses!) To multiply two matrices, say $A \times B = C$, you need a rule to calculate each element of C from the elements of A and B. Here it is:

$$C = AB \Longrightarrow C_{ij} = \sum_k A_{ik} B_{kj}.$$

Take a look at the pattern of indices. It alone tells you some properties of the multiplication:

1) The number of *columns* of A must be the same as the number of *rows* of B, otherwise the two matrices simply can't be multiplied.
2) The number of rows of C = the number of rows of A.
3) The number of columns of C = the number of columns of B.

Here is an example. We will take A as given above, and multiply it by B:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}; \qquad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}.$$

A is 3×3 and B is 3×2, so property 1) is satisfied, and properties 2) and 3) tell us that $C = AB$ is a 3×2 matrix. We use our multiplication rule to calculate each of the six elements of C. An example:

$$\begin{aligned} C_{12} = \sum_k^3 A_{1k} B_{k2} &= A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \\ &= 1 \times 2 + 2 \times 4 + 3 \times 6 = 28. \end{aligned}$$

Can you see that the sum over k *has* to be from 1 to 3? There is a way to do the bookkeeping without writing down all the steps in the sum. Note that what we are doing when we calculate C_{ij} is taking the "usual" vector dot product between the ith row of A and the jth column of B. I do this by running my left index finger along the appropriate row of A and my right index finger along the appropriate column of B. You can quickly develop the rhythm and coordination to do this. (Have your instructor show you this if my verbal description is inadequate.)

Here is the complete result, and another for you to practice on:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 22 & 28 \\ 49 & 64 \\ 76 & 100 \end{pmatrix}; \qquad \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 3 \end{pmatrix}.$$

One final point: note that matrix multiplication with our definition is not commutative, that is, $A \times B \neq B \times A$! With our examples, you can't even multiply them in the other order, let alone get the same answer. Even for square matrices with the same number of rows, which *can* be multiplied in either order, the answer won't in general be the same. Try it with a 2×2 example:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}; \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In this, Werner Heisenberg found the mathematical roots of the Uncertainty Principle.