Learning Guides

for

PHYSICS 103/105

Princeton University

Fall 2004

Princeton University
Physics 103
Fall Term 2004

Learning Guide No. 1: One-Dimensional Motion, Vectors, Motion in Two and Three Dimensions

Suggested Reading: Tipler, Chapters 1, 2, and 3

Problem I

A student is rowing a rowboat on a river whose current flows with speed v. She can row at a speed V with respect to the water. Since $V > v$ she can row upstream as well as downstream. She decides to row a distance d (relative to the shoreline) upstream and then turn around and row downstream the same distance d to her starting point.

1. How long does it take the student to complete the round trip? If your answer doesn't check even after a good effort, look at Helping Questions 1 and 2. **Key 10**
2. Does the trip take longer when $v = 0$ or when $v > 0$? Why? **Key 12**

Problem II

A stone is dropped from the roof of an 80-m-high building. Its displacement, y, measured from the point of release is given by $y(t) = 5t^2$, where y is in meters and the time t is in seconds. In this problem you will calculate the stone's acceleration in two ways: by averaging over short time intervals, and by using calculus.

1. What is the displacement of the stone at $t = 0, 1, 2, 3$, and 4 s? What are the average velocities in the intervals 0–1 s, 1–2 s, 2–3 s, and 3–4 s? For small intervals, the instantaneous velocity at the center of an interval is approximately equal to the average velocity over the interval. Assuming for sake of argument that these two quantities are equal, what is the average acceleration in the time intervals 0.5–1.5 s, 1.5–2.5 s, and 2.5–3.5 s? If you're having trouble, use Helping Question 3. **Key 25**
2. Use calculus (i.e., take a derivative) to find the instantaneous velocity as a function of time. Find the instantaneous acceleration as a function of time. If you are stuck, see Helping Question 4. **Key 21**
3. If the physical situation had been different so that the displacement function was $y(t) = 5t^4$ instead of $y(t) = 5t^2$, which method would have been more accurate for calculating the instantaneous acceleration? **Key 6**

Problem III

Vector Warm-Ups

1. Let $\mathbf{A} = 3\mathbf{i} + 4\mathbf{j}$ and let $\mathbf{B} = 5\mathbf{i} + 12\mathbf{j}$. What is the magnitude of $\mathbf{A}$? What is the magnitude of $\mathbf{B}$? What is $\mathbf{A} + \mathbf{B}$? What is the angle between $\mathbf{A}$ and $\mathbf{B}$? If you're having trouble, look at Tipler Sections 3-1 and 3-2. **Key 19**
2. Now let $\mathbf{A} = 3\mathbf{i} - 4\mathbf{j}$ and let $\mathbf{B} = -5\mathbf{i} - 12\mathbf{j}$. What is the magnitude of $\mathbf{A}$? What is the magnitude of $\mathbf{B}$? What is $\mathbf{A} + \mathbf{B}$? What is the angle between $\mathbf{A}$ and $\mathbf{B}$? **Key 9**
3. Show the region in the xy-plane that contains the end points of all possible vectors $\mathbf{A} + \mathbf{B}$ where $\mathbf{A}$ is a vector of magnitude 5 and $\mathbf{B}$ is a vector of magnitude 13. **Key 1**

Problem IV

An amateur football player punts a football straight up at $v_0 = 15$ m/s. Assume that the acceleration due to gravity is $g = 10$ m/s^2.

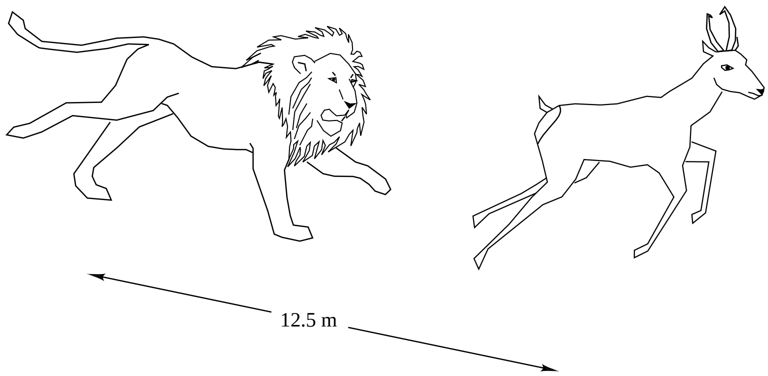
1. How long is the football in the air? If you need help getting started, use Helping Questions 5 and 6. **Key 26**
2. How high does the football go? Helping Question 7 will put you on the right track. **Key 7**

Next, a professional football player punts a football straight up. The football leaves at twice the speed of the amateur player's football; i.e., with $v_0 = 30$ m/s.

3. Does the ball stay in the air twice as long? **Key 11**
4. Does it go twice as high? **Key 3**

Problem V

A grazing antelope first notices a lion attacking when the lion is 12.5 m away and moving toward the antelope at a speed of 5 m/s. The antelope begins to accelerate away from the lion at 3 m/s^2 and the lion simultaneously begins to accelerate at 2 m/s^2.



1. How long does the antelope's flight last? **Key 15**

2. How far has the antelope traveled when the lion catches up with it? **Key 4**

If you need help see Helping Question 8.

Problem VI

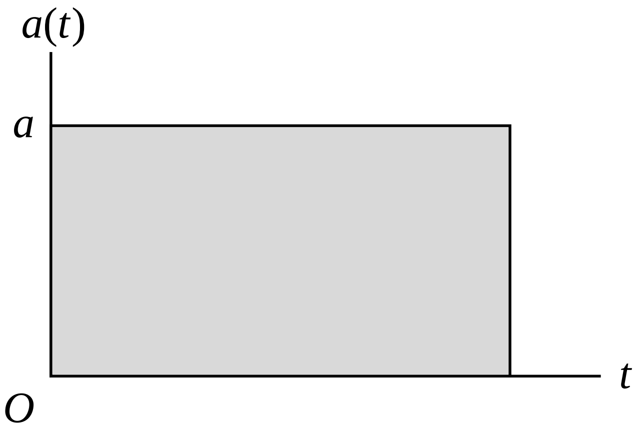
An object dropped from a stationary helicopter falls straight down toward the earth. At the beginning of its flight, its acceleration is g, the acceleration due to gravity. As its speed increases, it meets with increasing amounts of *air resistance*. To a reasonable approximation, its motion satisfies $a(t) = g - kv(t)$ where $a(t)$ and $v(t)$ are the instantaneous acceleration and velocity and k is a positive constant that depends on the shape and surface roughness of the object. *Without* solving an equation for $v(t)$ use physical reasoning to sketch a graph of speed versus time. You should discover the **terminal velocity** phenomenon: the object never accelerates past a particular speed, called the *terminal velocity*. What is the terminal velocity for the object in terms of g and k? Turn to Helping Questions 9 and 10 if you need help. **Key 5**

Problem VII

In Problem II, you were given $x(t)$ and asked to find $v(t)$ and $a(t)$. Suppose you were given $a(t)$ and asked to find $v(t)$ and then $x(t)$. In this problem, you will learn a geometrical solution to this question. You know that:

$$v(t) = v_0 + at$$

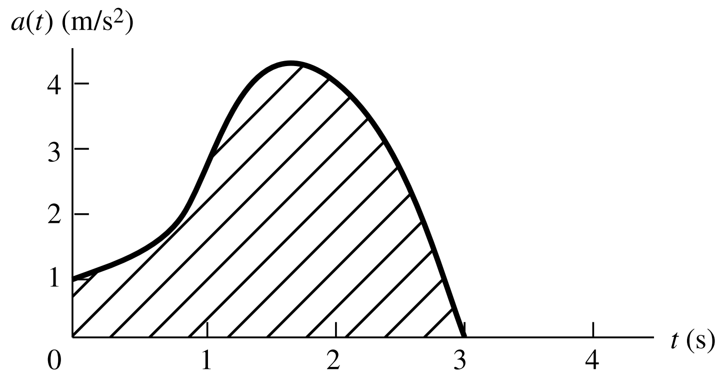
for motion with constant acceleration. Referring to the graph of $a(t)$ in the figure, note that:



$$v(t) = v_0 \ + \text{ area under the curve between 0 and } t, \tag{1}$$

since the shaded area is precisely at. In fact, it turns out that equation (1) is generally true, even if the acceleration is *not* constant. In such cases, the equation $v = v_0 + at$ no longer makes sense (since a is no longer a single number).

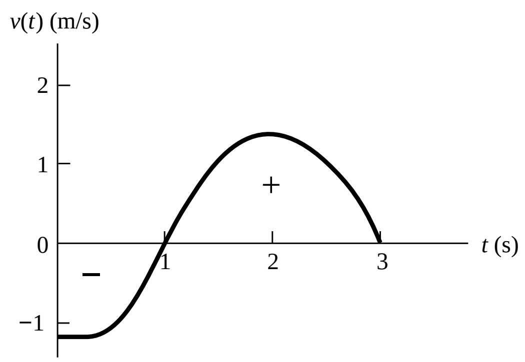
As an example, consider the next sketch, where $a(t)$ varies with time. The area of the shaded region is about 6 m/s, and so the velocity at $t = 3$ s is $v_0 + 6$ m/s.



In the same way, the equation of motion $x(t) = x_0 + vt$ not only applies for motion at constant v but generalizes to:

$$x(t) = x_0 \; + \; \text{area under the } v(t) \text{ curve between 0 and } t$$

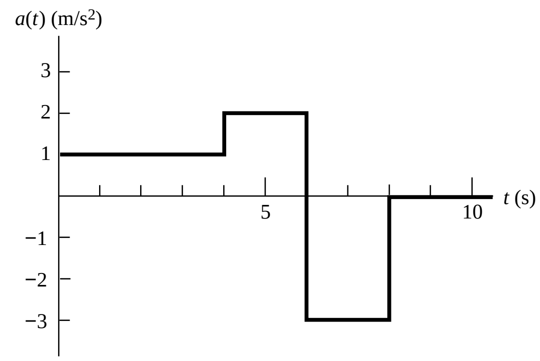
no matter how the velocity changes with time.



For both the acceleration and the velocity curves, a slight complication enters if the curves go beneath the horizontal axis — i.e., if the acceleration or velocity goes negative. Then "area under the curve" must be replaced by "area above the horizontal axis *minus* the area below the horizontal axis," as shown in the sketch. You should convince yourself that this extension is physically reasonable. In the example shown, the position at $t = 3$ s is about $x(3) = x_0 + 1$ m. (Think about where the particle is at the top of the first "+" bump.)

For practice, consider a particle moving with acceleration given by the adjacent graph. Take $v_0 = 0$ and $x_0 = 0$. Graph as accurately as you can, putting numbers on the axes:

1. $v(t)$ **Key 2**
2. $x(t)$ **Key 17**



Problem VIII

A student can row a boat 8 km/h in still water. He is on one bank of an 8-km-wide river that has a current of 4 km/h.

1. What is the smallest amount of time he needs to get to the other bank, assuming that it's acceptable to land anywhere on the other shore? If you're confused, see Helping Question 11. **Key 41**

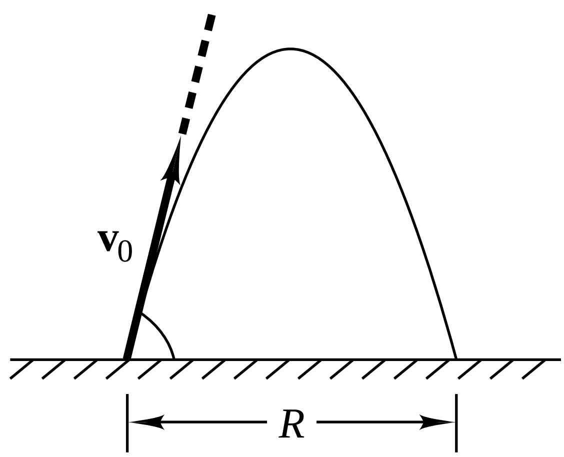
2. If the student wants to get to the point exactly across from where he starts, what angle should his boat make with respect to a line running straight across the river? Helping Question 12 will help you with this part. **Key 31**

3. Take the point of view of an observer on the shore. In your reference frame, the student's velocity vector is the sum of a vector of length 8 km/h (his rowing speed in still water) and a vector of length 4 km/h (the speed of the river's current). In your reference frame, what is the student's speed in part (1)? In part (2)? **Key 34**

If you need some hints, turn to Helping Questions 13 and 14.

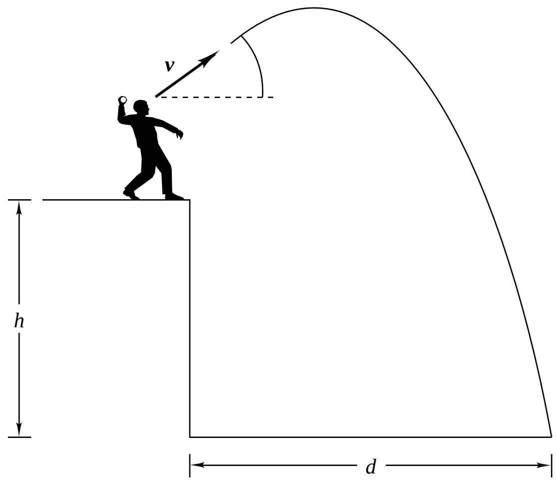
Problem IX

A cannon fires a projectile at an angle θ with respect to the horizontal. The speed of the projectile as it leaves the cannon's barrel is v_0. Find an expression that gives the horizontal range of the shell, R, in terms of θ, v_0, and g. See Helping Questions 15 and 16. (This is the **range formula**, which is also derived in Tipler, but don't look there unless you're really stuck.) **Key 40**



Problem X

A woman standing on a cliff of height h has a baseball that she can throw with speed v. She wants to throw the baseball as far away from the cliff as she can. In terms of the variables indicated on the diagram, she wants to choose θ to maximize d. For this problem you may neglect the height of the woman, air resistance and the bouncing or rolling of the ball. Before you start, you might want to guess at $\theta_{\rm max}$ for $h = 0$ and h very high. **Note:** The range formula from Problem II does *not* apply. It is valid only in cases where the initial and final heights are the same.



1. Express d in terms of v, θ, h, and g, the acceleration due to gravity. Use Helping Questions 17 and 18. **Key 46**

2. Set $h = 0$ in your answer for part (1). What is θ_{max}? Does θ_{max} depend on v? For a hint, see Helping Question 19. **Key 30**

3. Now suppose that h is "infinitely large" so that terms not containing h can be ignored. What is θ_{max}? Does θ_{max} depend on v? Stumped? See Helping Question 20. **Key 29**

4. Finally, consider the case where h is finite and positive. Between what two values of θ is θ_{max}? Use your physical intuition to decide whether θ_{max} dependes on v. Use Helping Question 21 for the last question. **Key 43**

HELPING QUESTIONS

1. What is the student's speed relative to the shoreline when she rows upstream? When she rows downstream? **Key 24**
2. How long does it take her to row upstream? Downstream? **Key 8**
3. What is the definition of average velocity? Average acceleration? **Key 13**
4. Define instantaneous velocity and acceleration mathematically. **Key 20**
5. Can you think of a kinematic formula that relates what you know: the starting point $y_0 = 0$, the end point $y = 0, v_0$, and g; to what you want to find — the time of landing t? **Key 22**
6. What are the two roots of the equation $v_0 t - \frac{1}{2}gt^2 = 0$? **Key 14**
7. You know the time at which the ball hits the ground. Can you say right away the time at which the ball is at the high point of the path? If you don't see the intuitive answer, again try to find the right equation. The key step is to think of what quantity has a special behavior at the top. **Key 18**
8. What is true about the two animals' positions at the time of interest? **Key 27**
9. Is the velocity of the object increasing or decreasing? Is the acceleration of the object increasing or decreasing? What do the answers to these two questions mean in terms of the plot of $v(t)$? **Key 16**
10. Suppose the object was falling with terminal velocity. What would the acceleration be? **Key 23**
11. Does the speed of the river's current affect the crossing time? **Key 33**
12. What direction must the velocity vector (rowing velocity plus current velocity) be pointing? Make a sketch, showing all three vectors and the angle you're after. **Key 39**
13. One way you can add vectors is to introduce a coordinate system and add the vectors by component. What would be a convenient coordinate system here? **Key 32**
14. If a vector $\mathbf{v}$ makes an angle θ with the x-axis, what is v_x in terms of $|\mathbf{v}|$ and θ? What is v_y? **Key 37**
15. What is $x(t)$? What is $y(t)$? If you need another hint, move on to Helping Question 6. **Key 44**
16. $y(t)$ has two zeros, what is their significance? **Key 28**
17. If you knew the time t the ball spends in the air, could you get d in terms of θ, v, and t? **Key 42**
18. Can you get the time of flight t from the vertical component of the motion? You'll have to use the quadratic formula. **Key 35**
19. A trigonometric identity says that $2\sin\theta\cos\theta = \sin(2\theta)$. Can you get the maximum by thinking about the graph of $\sin(2\theta)$? Alternatively, you can use calculus, setting the

derivative of d with respect to θ equal to zero. **Key 38**

20. What is the expression for d when the terms not containing h are ignored? **Key 36**

21. Think of a child and a professional baseball player on a 5-m cliff. The baseball player has a much higher v of course. To the child, is the cliff extremely high or negligible? What about to the baseball player? **Key 45**

Notes: Dimensions

You should read Chapter 1 of Tipler about units and dimensions very carefully. As this course progresses, you will appreciate more and more their statement that *the dimensions on one side of an equation must be the same as those on the other side.* In fact, in any legitimate physical equation the dimensions of *all the terms* must be the same. Perhaps more to the point, a look at the dimensions of a solution to a problem often provides a quick "sanity check" on its veracity. If, for example, you are asked to calculate a velocity or a speed, your answer had better have units of meters per second. Any other dimensions indicate that something has gone awry. Old pros employ this trick to check their work as standard practice.

One can even check dimensions in equations that contain derivatives. The dimensions of a derivative are identical to the dimensions of the corresponding fractions. Thus, for example, $\mathrm{d}x/\mathrm{d}t$ has the same dimensions as x/t. Later on you will learn how to use integrals in physics, but from the point of view of dimensions an integral is just like a multiplication. You will also become familiar with algebraic operations between vectors — vector addition and two different types of vector multiplication. Dimensionally, these operations are the same as normal addition and multiplication.

One can even use dimensional analysis to make "ballpark" estimates. To illustrate this, let's try to solve Problem IV about punted footballs using only dimensional arguments and intuition. Part (1) asks for the time t that the ball is in the air. Intuitively, the greater the initial speed v_0, the greater the time of flight t. Intuition also says that the greater gravity g is, the smaller t is. So a possible solution is:

$$t = \frac{v_0}{g} \qquad \left(\frac{[L]/[T]}{[L]/[T]^2} = \frac{1}{1/[T]} = [T] \right)$$

This equation checks dimensionally, as the computation in the parentheses shows (in that computation, $[L]$ and $[T]$ denote quantities having dimensions of length and time, respectively). In fact, the exact solution is $t = 2v_0/g$. This is about as close as dimensional analysis can get since the "2" is dimensionless. This sort of result is typical of the kind of accuracy one can expect using this approach.

Part (2) asks for the maximum height h of the ball. Again, intuition suggests a possible solution:

$$h = \frac{v_0}{g} \qquad \left(\frac{[L]/[T]}{[L]/[T]^2} = [T] \neq [L] \right).$$

But this equation does not check dimensionally since $[L] \neq [T]$. If you think about it, the only simple way to fix it up is to write:

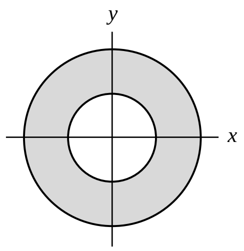
$$h = \frac{v_0^2}{g} \qquad\qquad \left(\frac{([L]/[T])^2}{[L]/[T]^2} = \frac{[L]^2/[T]^2}{[L]/[T]^2} = [L] \right),$$

which does check dimensionally. Again, it is off from the exact expression, $h = v_0^2/2g$, by a dimensionless factor of 2.

The real triumph of dimensional analysis in this example is the answers it gives for parts (3) and (4). Even though it is off by dimensionless factors, it still predicts correctly that if the initial speed is doubled, the time of flight is also doubled but the maximum height is multiplied by 4.

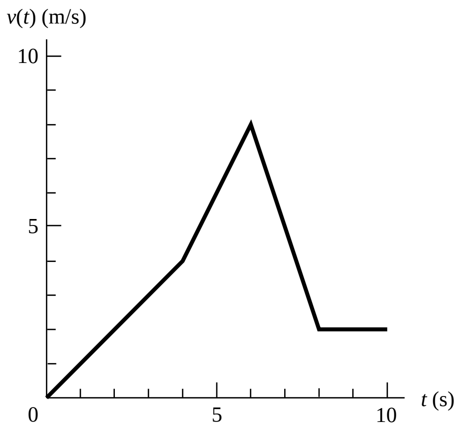
ANSWER KEY

1.



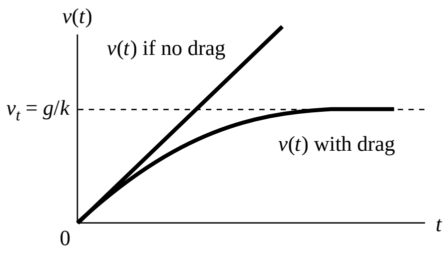
Inner circle has radius 8
Outer circle has radius 18

2.

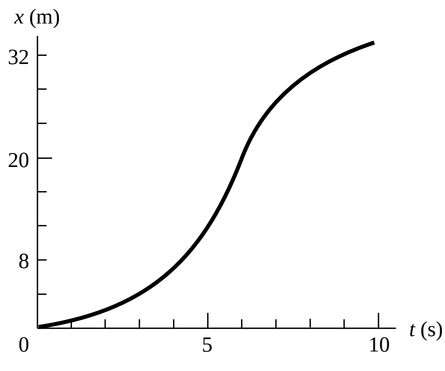


3. No, it goes 4 times higher.
4. 37.5 m

5. $v_t = g/k$



6. The calculus method of part (2) would have been better. No matter how small the time intervals were made in part (1), the answer would still be a little off.
7. 11.25 m
8. $d/(V - v)$ upstream;
   $d/(V + v)$ downstream
9. $|\mathbf{A}| = 5$, $|\mathbf{B}| = 13$, $\mathbf{A} + \mathbf{B} = -2\mathbf{i} - 16\mathbf{j}$, $\theta_{\mathbf{AB}} = 59.5°$
10.
$$\frac{d}{V - v} + \frac{d}{V + v} = \frac{2dV}{V^2 - v^2}$$
11. Yes
12. The trip takes longer when $v > 0$ because when $v > 0$ the student's average speed is less than V: she spends more than half the time at speed $V - v$ and less than half the time at speed $V + v$.
13.
$$v_{\rm av} = \frac{\Delta y}{\Delta t}; \qquad a_{\rm av} = \frac{\Delta v}{\Delta t}$$
14. The starting time $t = 0$ and the landing time
$$t = \frac{2v_0}{g}.$$
15. 5 s
16. The speed is increasing but the acceleration is decreasing, so the slope of the speed curve will always be positive but will become less steep at larger t.
17.



18. Intuitive method: It's at the top at the midpoint of the trip, or $t = 1.5$ s. Systematic method: The key point is that the ball stops instantaneously at the top, i.e., $v = 0$. Use $0 = v_0 - gt$ to solve for the time to get to the top.

19. $|\mathbf{A}| = 5$, $|\mathbf{B}| = 13$, $\mathbf{A} + \mathbf{B} = 8\mathbf{i} + 16\mathbf{j}$, $\theta_{\mathbf{AB}} = 14.3°$

20.
$$v = \frac{dy}{dt}; \quad a = \frac{dv}{dt}$$

21. $v = 10t$, $a = 10$, so the acceleration is constant at 10 m/s^2.

22. $y = y_0 + v_0 t + \frac{1}{2}at^2; \quad a = -g$

23. 0

24. $V - v$ upstream; $V + v$ downstream.

25. For $t = 0, 1, 2, 3, 4$ s:
$y = 0, 5, 20, 45, 80$ m
$v_{\text{av}} = 5, 15, 25, 35$ m/s
$a_{\text{av}} = 10, 10, 10$ m/s^2

26. 3 s

27. They are equal, i.e. at the time of interest $x_{\text{lion}}(t) = x_{\text{antelope}}(t)$

28. Launch time and landing time

29. $\theta_{\max} = 0°$, independent of v.

30. $\theta_{\max} = 45°$, independent of v.

31. 30° upstream

32. Let the x-direction be downstream and the y-direction be across the river.

33. No

34. $\sqrt{80} \simeq 8.94$ km/h for part (1); $\sqrt{48} \simeq 6.93$ km/h for part (2).

35.
$$t = \left(\frac{v \sin\theta}{g} + \frac{1}{g}\sqrt{v^2 \sin^2\theta + 2gh} \right)$$

36.
$$d = v \cos\theta \sqrt{\frac{2h}{g}}$$

37. $v_x = |\mathbf{v}| \cos\theta$; $v_y = |\mathbf{v}| \sin\theta$

38. $\sin(2\theta)$ has a maximum at $\theta = 45°$.

39. Straight across the river

40.
$$R = \frac{2v_0^2}{g} \sin\theta \cos\theta = \frac{v_0^2}{g} \sin(2\theta)$$

41. 1 h

42. $d = (v \cos\theta)\, t$

43. $\theta_{\max}$ is between 0° and 45° and depends on v.

44. $x(t) = (v_0 \cos\theta)\, t$
$y(t) = (v_0 \sin\theta)\, t - \frac{1}{2}gt^2$

45. The cliff is high to the child, and negligible to the player.

46. $d = v \cos\theta\, \times$
$$\left(v \frac{\sin\theta}{g} + \sqrt{\frac{v^2 \sin^2\theta}{g^2} + \frac{2h}{g}} \right)$$

Princeton University
Physics 103
Fall Term 2004

Learning Guide No. 2: Newton's Laws

Suggested Reading: Tipler, Chapters 4 and 5 (Optional: Section 5-4)

Problem I

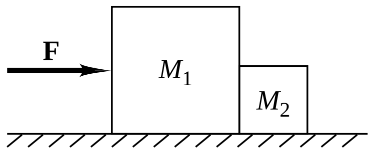
A 100-kg person is standing on a bathroom scale in an elevator. Take the acceleration g due to gravity to be $g = 10$ m/s^2.

1. Suppose that the elevator is moving at a constant speed of 2 m/s. What is the reading on the scale? (The scale is calibrated in kilograms, not in newtons). **Key 36**

2. Suppose that the elevator is accelerating up at the constant rate of 2 m/s^2. What does the scale read? **Key 1**

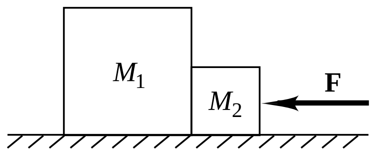
Problem II

Two blocks, in contact on a frictionless table, are pushed by a horizontal force applied to one block, as shown in the figure.



1. What is the force of contact between the two blocks in terms of F, M_1, and M_2? Answering Helping Question 1 will get you started. **Key 19**

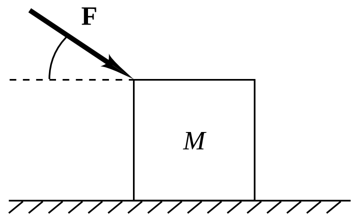
In a second situation, a force of equal magnitude but opposite direction is applied to the other block.



2. What is the force of contact in this situation? If $M_1 \neq M_2$, is the force the same as in part (1)? **Key 33**

Problem III

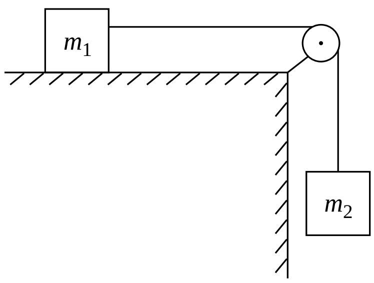
A block of mass M lies on a horizontal surface. A force $\mathbf{F}$ acts on the block at an angle θ from the horizontal. The coefficient of static friction is μ_s.



1. What is the minimum force necessary to start the block moving as a function of the angle θ? Use Helping Questions 2, 3, and 4 if necessary. **Key 20**
2. From your answer to part (1), can you find a critical angle θ_c such that for angles θ greater than θ_c even huge forces won't move the block? If $\mu_s = 0.4$, what is θ_c? Turn to Helping Question 5 if you're confused. **Key 8**
3. Think physically about why there is a critical angle past which the block won't slide. What's increasing with θ and what's decreasing as θ increases? **Key 34**

Problem IV

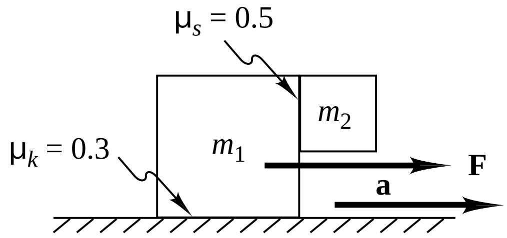
In the apparatus shown in the sketch, the string is massless and does not stretch, and the pulley is massless and has frictionless bearings. The coefficient of kinetic friction between block 1 and the table is μ_k. Assuming that M_2 is large enough that the blocks move, find the acceleration of the blocks. Use Helping Questions 6, 7 and 8, if you need to. **Key 39**



Problem V

A man driving a car 60 mi/h down a country road sees that a huge tree has fallen across the road some distance in front of him. He immediately slams on his brakes and skids to a stop. Fortunately the pavement is dry and the coefficient of kinetic friction between the tires and the road is $\mu_k = 0.5$. How many feet does it take the car to stop? (60 mi/h equals 88 ft/s.) Use $g = 32$ ft/s^2. If you're really stuck, use Helping Questions 9 and 10 sparingly. **Key 42**

Problem VI



Two blocks of mass $m_1 = 10$ kg and $m_2 = 2$ kg slide along a horizontal surface. An external force $\mathbf{F}$ applied to m_1 from the left provides enough acceleration to keep m_2 from sliding down the face of m_1. The coefficient of *kinetic* friction between m_1 and the horizontal surface is $\mu_k = 0.3$ and the coefficient of *static* friction between m_1 and m_2 is $\mu_s = 0.5$. Take g to be 10 m/s^2.

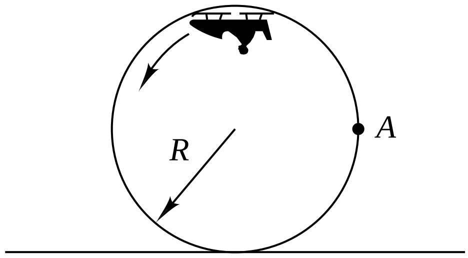
1. Assuming that m_2 does not slip, find a, the acceleration of the two blocks when $F = 350$ N. If need be, see Helping Question 11. **Key 40**
2. For $F = 350$ N, what are the components of $\mathbf{F}_{21}$, the force exerted on m_2 by m_1? See Helping Question 12. **Key 14**
3. What is the smallest value of F required to keep m_2 from slipping ? See Helping Question 13. **Key 18**

Problem VII

Observers in an elevator can't distinguish between their own acceleration and the earth's gravity. In the same way, the acceleration caused by uniform circular motion feels like gravity. On the earth's surface the acceleration due to gravity is, to three significant figures, 981 cm/s^2. But if experimenters at the equator measured the acceleration of objects falling in a vacuum, they would get a number less than this because of the earth's rotation.

1. To three significant figures, what acceleration would be measured? (The equatorial radius of the earth is about 6000 km.) See Helping Question 14, if you need it. **Key 7**
2. Why were you able to use such a crude approximation for the earth's radius when you wanted three-figure accuracy? **Key 15**

Problem VIII

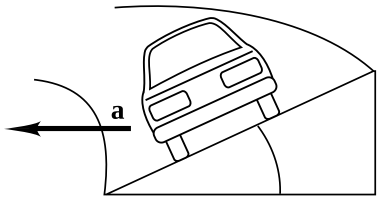


A stunt person rides a bobsled through a loop-the-loop track as shown in the diagram. The loop has radius R.

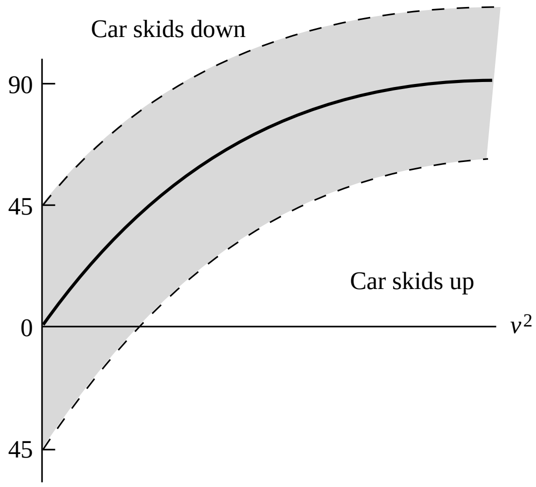
1. Find the minimum speed v_t at the top of the track that ensures that the sled will stay on the track. See Helping Questions 15 and 16. **Key 4**

2. On a second pass through the loop, the sled has speed v_A when it reaches point A on the loop. Find the magnitude and direction of the *net* force acting on the sled at point A. For help, see Helping Questions 17 and 18. **Key 27**

Problem IX



An engineer designing the bank on a curved road has to take friction into account. There are a lot of variables, so she might use a graph such as the one below to organize her thoughts. For this graph the radius of the curve R and the coefficient of friction μ_s are fixed. The solid line is the "perfect" bank: the car goes around the curve and the road exerts *no sideways frictional force* on the car. The dashed lines denote the steepest possible angle and the gentlest possible bank angles θ that allow the car to negotiate the curve without skidding sideways. In parts (1) to (3) of this problem, you will get equations for these three curves.

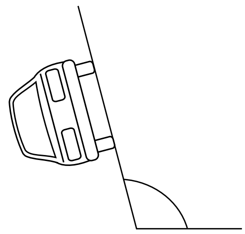


1. For what speed v is the angle θ a perfect bank? Express v in terms of θ, g, and R. Helping Questions 19 and 20 will give you the method of attack, but by now you should know it. **Key 13**

2. At what speed v will the car just begin to slide *up* a bank of angle θ? Express v in terms of θ, g, R, and μ_s. Helping Questions 21 and 22. **Key 3**

3. At what speed v will the car just begin to slide *down* a bank of angle θ? Write v in terms of θ, g, R, and μ_s. Helping Question 23. **Key 22**

4. The diagram on the right shows a curve banked *past vertical* so that $\theta > 90°$. Referring to your answer for part (3), if $\mu_s > 0$, is there a speed great enough so that the car stays on the road? For a hint, see Helping Question 24. **Key 30**



5. The graph above has a certain *symmetry*: if for a speed v the perfect bank is θ and the car will just begin to slide down at $\theta + \Delta\theta$, then, the way the graph is drawn, it will just begin to skid up at $\theta - \Delta\theta$. Is this correct? The equations from parts (2) and (3) are a little messy, so try using physical reasoning. If need be, see Helping Question 25. **Key 32**

HELPING QUESTIONS

1. What is the acceleration of the two blocks? The force on the right block? **Key 16**
2. Draw a force (free-body) diagram and label all the forces. What is the frictional force when the applied force F is at its maximum value before slipping? **Key 26**
3. What is the acceleration a when the applied force F is at its maximum value — i.e. just before the block starts slipping? **Key 6**
4. What is the horizontal component of Newton's second law $\mathbf{F} = m\mathbf{a}$? The vertical component? **Key 37**
5. As θ increases from zero, the minimum force F required also increases. For what value of θ is the minimum force required infinite? **Key 29**
6. Draw a force (free-body) diagram for each block and apply $\mathbf{F} = m\mathbf{a}$ to each block. Let T_1 be the force of the string on block 1 and T_2 its force on block 2. Let a_1 be the acceleration of block 1 and a_2 the acceleration of block 2. **Key 5**
7. Is there a relation between T_1 and T_2? **Key 31**
8. Is there a relation between a_1 and a_2? **Key 43**
9. Draw a force (free-body) diagram and find the resultant force on the car. **Key 41**
10. Is it possible reduce the problem to a kinematics problem with constant acceleration? **Key 12**
11. If m_1 doesn't slip is there any difference between this situation and the case of a single block with mass $m_1 + m_2$? **Key 2**
12. What *other* force or forces act on m_2 ? Given these, Newton's second law, and the acceleration from part (1), one can deduce the components of $\mathbf{F}_{21}$. **Key 23**
13. What is the physical significance of each component of $\mathbf{F}_{21}$? **Key 10**
14. You know that the earth rotates once every 24 h. What is the acceleration v^2/R of a point on the equator? **Key 9**
15. What forces act on the sled? What are the magnitude and direction of its acceleration? **Key 17**
16. What is the normal force just at the point when the sled is about to fall? **Key 11**
17. What is the force in the vertical direction? **Key 21**
18. What equation expresses Newton's second law in the horizontal direction? **Key 24**
19. Draw a force diagram for the car and identify the acceleration vector. **Key 28**

20. What is the horizontal component of $\mathbf{F} = m\mathbf{a}$? The vertical component? **Key 38**

21. The friction force is parallel to the road. If the car is about to slide up the road, does the friction force point up or down the bank? What is its magnitude in terms N, the normal force? **Key 44**

22. So now what are the components of $\mathbf{F} = m\mathbf{a}$? **Key 45**

23. With the car about to slide down the road, does the friction force point up or down the bank? What is its magnitude in terms of N, the normal force? Now that you have the sign right, the rest of the solution proceeds the same way as for the car sliding up the bank. **Key 46**

24. When does the denominator of the answer to part (3) equal zero? **Key 25**

25. An observer in the car can't distinguish between gravity and his own acceleration. Imagine that he decides that he is not accelerating but rather that the force from gravity is greater than usual and angled strangely. How would he draw the force diagram for the car when the bank is perfect? **Key 35**

Notes: More About Dimensional Analysis

Mentally Varying Variables

As you can see by now, a big part of learning physics is learning how to manipulate and understand equations. The first and most important thing to learn is manipulate the equations while they are still in variable form and only plug in the numbers at the end. Indeed, the ability to solve problems using algebra allows us to extend our mental "grasp" in a very profound way. Learn to take advantage of algebra, one of humanity's great achievements!

This Learning Guide has tried to force you to acquire this habit by not always giving you numbers. There are at least two advantages of not plugging numbers in. The first relates to dimensional analysis. If you choose to plug in the numbers from the start of a problem, you will quickly lose track of the dimensions unless you are *extremely* careful (in which case you'd be taking our advice!).

Indeed, you don't need to wait until the end of a problem before making dimensional checks. Certain quantities are always dimensionless. For example, whenever you see $\sin x$, $\cos x$, or any trigonometric function, *its argument (x) must be dimensionless.* This is because if you changed units from, say SI to the British system, the angle must not change (angles are always the ratio of two like-dimensioned quantities). Also, the *value of any trigonometric function is dimensionless* (such quantities are even more obviously the ratio of like dimensioned quantities).

Another advantage has to do with *mentally varying the variables in any equation.* This not only provides another type of "sanity check" but is an excellent way to understand what an equation really means. As an example, let's try to fully understand the solution to Problem IV, with the two blocks and the pulley,

$$a = g\left(\frac{m_2 - \mu_k m_1}{m_2 + m_1}\right).$$

First, the equation is dimensionally correct since a and g are both accelerations, and the factor in parentheses is dimensionless (coefficients of friction, being the ratio of two forces, are dimensionless). Now imagine that m_1 was very small or even zero. Then

$$a = g\frac{m_2}{m_2} = g,$$

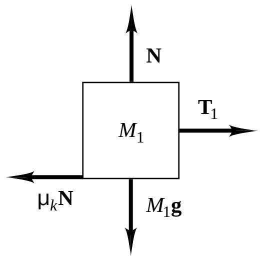
which you recognize as the acceleration of m_2 freely falling. Imagine instead that m_1 was very big, so big that m_2 was negligible. Then

$$a = g\left(\frac{-\mu_k m_1}{m_1}\right) = -g\mu_k.$$

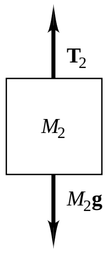
This equation is clearly *incorrect* because the masses certainly don't accelerate backward: they just sit there if m_1 is too massive. If you think about it, you have just discovered the critical condition for acceleration: m_2 must be greater than $\mu_k m_1$. For practice, further your understanding of this equation by seeing what happens to a when you (1) vary g, (2) vary μ_k, or (3) double m_1 and m_2 simultaneously. Your answers should match your physical intuition. Of course, if you just started with something like $a = 3.7$ m/s^2, you couldn't do any of this.

This method of mentally varying variables to extremes is also a good way to get a feeling for a problem before you actually begin writing equations. For Problem III, about pushing the block hard enough to overcome static friction, if you let θ be 90° and push straight down then, of course, the block doesn't move! So, it's not too surprising that there is a critical angle near 90°. For Problem II, with two blocks in contact on a table, take one of them to be huge — say, your physics textbook — and one of them to be tiny — perhaps a little ball of clay. Of course, the forces of contact are different — if you push your text with a little ball of clay, the ball immediately squishes, but if you push the ball with your text the ball just rolls away!

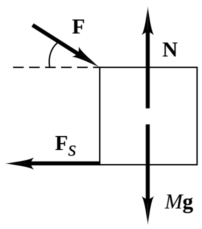
ANSWER KEY

1. 120 kg
2. No
3. $$v = \sqrt{gR\left(\frac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right)}$$
4. $v_t = \sqrt{gR}$
5. 

so $N = M_1 g$
and $T_1 - \mu_k N = M_1 a_1$



so $M_2 g - T_2 = M_2 a_2$

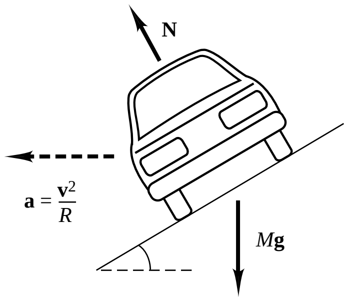
6. 0
7. 978 cm/s^2
8. $\theta_c = \cot^{-1}\mu_s = 68°$
9. 3 cm/s^2
10. F_{21x} is the normal force, F_{21y} is the force of static friction.
11. 0 (N can't be negative)
12. Yes, the constant acceleration is $\mu_k g = -16$ ft/s^2.
13. $$v = \sqrt{gR\,\tan\theta}$$
14. $F_{21x} = m_2 a = 52.3$ N;
$F_{21y} = m_2 g = 20$ N
15. The correction due to the earth's rotation is small, contributing only to the third decimal place.
16. Both blocks have acceleration $F/(m_1+m_2)$. The force on the right block is $m_2F/(m_1+m_2)$.
17. Gravity and the normal force, both straight down. $a = v_t^2/R$ inward (i.e., straight down).
18. $$F = \left(\frac{1}{\mu_s} + \mu_k\right)(m_1+m_2)\,g = 276 \text{ N}$$
19. Force of contact is
$$F_{\text{contact}} = F\left(\frac{M_2}{M_1+M_2}\right).$$
20. Minimum force for movement is
$$F_{\text{min}} = \frac{\mu_s Mg}{\cos\theta - \mu_s\sin\theta}.$$
21. mg (down)
22. $$v = \sqrt{gR\left(\frac{\sin\theta - \mu_s\cos\theta}{\cos\theta + \mu_s\sin\theta}\right)}$$
23. $m_2 g$ (acting down)
24. $N = mv^2/R$, where N is radially inward.
25. When $\cot\theta = -\mu_s$
26. 

At the maximum value of F before slipping, $F_s = \mu_s N$.

27. $$\left|\mathbf{N} + m\mathbf{g}\right| = m\sqrt{\frac{v_A^4}{R^2} + g^2}$$

$$\theta = \tan^{-1}\left(\frac{gR}{v_A^2}\right) \text{ below the horizontal.}$$

28.

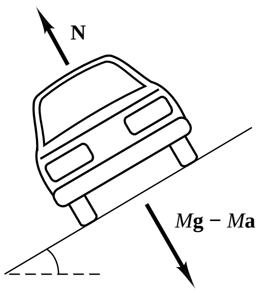


29. When the denominator is zero.
30. Yes, there is a speed high enough to keep the car on the road if $\cot\theta > -\mu_s$. Otherwise the car will slide down or fall off.
31. $T_1 = T_2$ (The pulley has no mass to accelerate.)
32. Yes
33.

$$F_{\text{contact}} = F\left(\frac{M_1}{M_1 + M_2}\right),$$

which is different from part (1).

34. Normal force, and thus the frictional force, increases with θ. The horizontal component of **F** decreases as θ increases. Both effects discourage sliding.
35.



36. 100 kg
37. Horizontal component:

$$F\cos\theta - \mu_s N = 0$$

Vertical component:

$$mg + F\sin\theta - N = 0$$

38.

$$\text{Horizontal}: \quad N\sin\theta = \frac{mv^2}{R}$$

$$\text{Vertical}: \quad N\cos\theta = mg$$

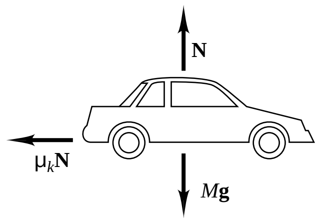
39.

$$a = g\left(\frac{m_2 - \mu_k m_1}{m_1 + m_2}\right)$$

40.

$$a = \frac{F}{m_1 + m_2} - \mu_k g = 26.2 \text{ m/s}^2$$

41.



so resultant force is $\mu_k mg$, slowing the car down.

42. 242 ft
43. $a_1 = a_2$ (the blocks are tied together and the string doesn't stretch).
44. Down the bank; $f = \mu_s N$
45.

$$\text{Horizontal}: \quad N\sin\theta + f\cos\theta = \frac{mv^2}{R}$$

$$\text{Vertical}: \quad N\cos\theta - f\sin\theta = mg$$

$$f = \mu_s N$$

46. Up the bank; $f = \mu_s N$

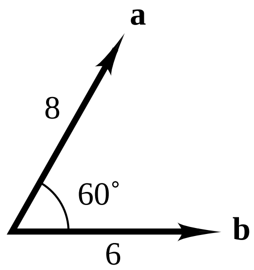
Princeton University
Physics 103
Fall Term 2004

Learning Guide No. 3: Work and Energy

Suggested Reading: Tipler, Chapters 6 and 7 (Optional: Sections 7-3,7-4)

Vector Warm-Ups — The Scalar Product

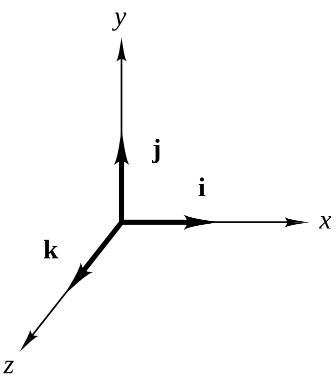
1. What is $\mathbf{a}\cdot\mathbf{b}$? If you don't remember what the definition of the scalar product is, go back to Tipler, Sec. 6-2. **Key 25**



There is another method of calculating scalar products that is occasionally more convenient than the formula $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$. You will discover this method in parts (2) and (3) and then use it in part (4).

2. $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors in the coordinate system drawn in the diagram. Fill in the multiplication table:

•	**i**	**j**	**k**
i			
j			
k			



If you need help turn to Helping Question 1. **Key 6**

3. In a certain fixed coordinate system, $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $\mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$. In terms of the components a_x, b_x, a_y, b_y, a_z, and b_z, what is $\mathbf{a}\cdot\mathbf{b}$? Stuck? Turn to Helping Question 2. **Key 18**

4. Suppose the vectors in part (1) are now given in terms of their components in a rectangular coordinate system. What is $\mathbf{a}\cdot\mathbf{b}$? **Key 32**

Problem I

A stone is released from a height h at time $t = 0$ and falls straight down under the influence of gravity. As you know, its height y at time t is given by $y(t) = h - gt^2/2$.

1. How much work W has the force of gravity done on the stone from the time of its release to time t? If you need help, use Helping Question 3. **Key 17**

As you also know, the stone's speed v at time t is given by $v = -gt$.

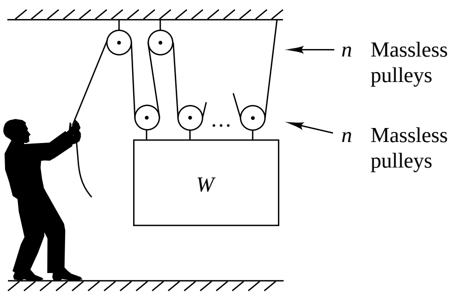
2. From the definition of kinetic energy, what is the kinetic energy K_0 at the start, and what is the kinetic energy K at time t? **Key 31**

3. Verify the work-energy theorem $W = \Delta K$ for this system. **Key 7**

4. From the definition of instantaneous power $P = \mathrm{d}W/\mathrm{d}t$, what is the power delivered by gravity to the stone at time t? From the formula $P = \mathbf{F} \cdot \mathbf{v}$, what is the power delivered by gravity to the stone at time t? Do your two answers agree? **Key 38**

Problem II

A construction worker wants to lift a block of weight $W = 1000$ lb a height $h = 5$ ft off the ground. He attempts to contrive a way to avoid doing 5000 ft·lb of work on the block. One idea he thinks up is to employ $2n$ pulleys, arranged as shown in the sketch. Assume that the tension is the same everywhere in the rope and that friction and the rope's deviation from the vertical can be neglected.



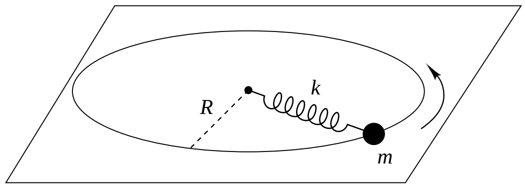
1. Using a force (free-body) diagram, determine the magnitude of the force F the worker needs to exert to lift the block in terms of W and n. Assume that the initial acceleration of the block is negligible and that once in motion the block moves with constant speed. Use Helping Questions 4 and 5 if you need to. **Key 23**

2. Through what distance d does the worker need to pull the rope in order to lift the block h ft? Use Helping Question 6, if necessary. **Key 5**

3. Does the contraption decrease the amount of work he has to do? **Key 8**

Problem III

A mass m is attached to a massless spring (with an unstretched length of l_0, and spring constant k) and moves in a circular path of radius R. Assume at first that there is no friction between the mass and the horizontal surface.



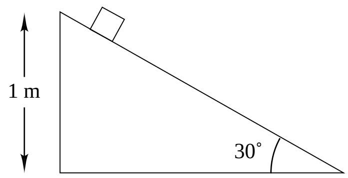
1. Find the ratio of the potential energy in the spring to the kinetic energy of the mass in terms of known quantities — i.e., (m, l_0, k, R, g). After a good try, use Helping Question 7. **Key 1**

2. Can the spring's potential energy equal or exceed the mass's kinetic energy? **Key 26**

3. Now, suppose a demon suddenly "turns on friction" between the mass and the surface. Find the distance d that the mass moves before it stops. Assume that the coefficients of static and kinetic friction are both very small and are a constant $= \mu$. Also assume that the final length of the spring is l_0. (Actually, it will be somewhat longer that l_0, but if μ is small, this can be ignored.) Stuck? Look at Helping Questions 8 and 9. **Key 30**

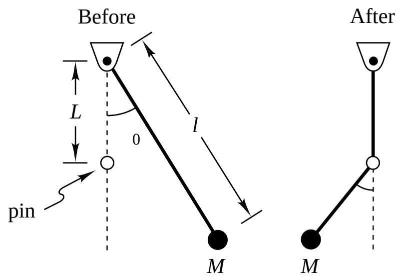
Problem IV

A block slides down a 1-m-high ramp, which is tilted at 30°. The coefficient of kinetic friction between the block and the ramp is $\mu_k = 0.4$. What is the block's speed at the bottom of the ramp? Stuck? Use Helping Question 10 and then Helping Question 11, if you need to. **Key 40**



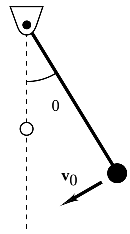
Problem V

A simple pendulum consisting of a mass M at the end of a string of length l is released from rest at an angle θ_0. A pin is located a distance $L < l$ directly below the pivot point.



1. After the string hits the pin, what is the maximum angle α that the pendulum makes with respect to the vertical? Stuck? Use Helping Question 12 and then Helping Question 13, if you need to. **Key 19**

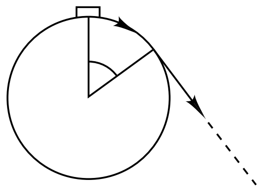
2. If, instead, the mass is given an initial velocity $\mathbf{v}_0$ as sketched here, what is the maximum angle α after the string hits the pin? **Key 12**



3. How does the answer to part (2) change if $\mathbf{v}_0$ is in the direction opposite to that shown in the figure? **Key 41**

Problem VI

A small mass m starts from rest and slides from the top of a fixed sphere of radius r.



1. If the sphere is frictionless at what angle θ from the vertical does the mass leave the surface? If you need a hint, use Helping Question 14. **Key 36**

2. Suppose there is a finite friction of $\mu_s = 0.1$ between the mass and the sphere. What is the minimum angle θ_{min} at which the mass will start to slide along the sphere? **Key 46**

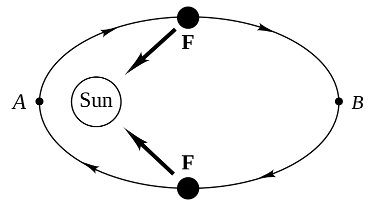
3. The mass is now placed just past this minimum angle and released. The coefficient of kinetic friction μ_k is small but non zero. Does the mass fly off at a larger or a smaller θ than was found in part (1)? Assume that in both cases θ is defined with respect to the top of the sphere. **Key 14**

Problem VII

The power of an automobile engine is usually measured in horsepower (hp) instead of watts (W). One horsepower equals 746 W. A typical automobile engine can sustain an output of 100 hp for a long period of time. Find some clever way to estimate how much sustained power *you* can put out, say, for half an hour. Need an idea? Try Helping Question 15. **Key 9**

Problem VIII

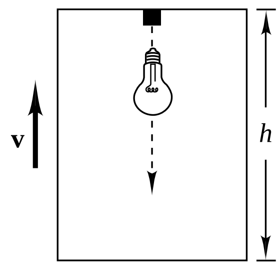
The gravitational force exerted on a planet by the sun is attractive, so the planet's potential energy is greater the farther from the sun it is. The planet moves in slightly elliptical orbit, as indicated in the diagram.



1. Is the planet moving faster at point A or point B? If you can't figure this out, turn to Helping Question 16. **Key 10**
2. During what part of the orbit is the sun doing positive work on the planet? Negative work? See Helping Question 17, if you're stuck here. **Key 3**
3. Check that when the sun is doing positive work the power $\mathbf{F}\cdot\mathbf{v}$ is positive; see Helping Question 18. **Key 15**

Problem IX

The work done by a force depends on the reference frame from which the system is observed. The kinetic energy of a particle also depends on the reference frame. However, in all inertial reference frames the work-energy theorem $W = \Delta K$ holds. In this problem, you will verify these three statements for a sample system viewed from two different reference frames.



An elevator of height h moves upward at constant velocity v. Vibrations cause the elevator's light bulb to fall from its fixture at time $t = 0$. The bulb hits the floor at time t later. Call the mass of the bulb m.

1. One observer views the bulb from the elevator. In this frame, what is the initial kinetic energy K_0? The final kinetic energy K? The work W done by gravity? Since you are trying to verify the work-energy theorem, do not use any energy principles you have learned: calculate the work done from the definition $W = Fd$. **Key 11**

2. A second observer views the bulb from the ground. In this frame, what is the initial kinetic energy K_0? The final kinetic energy K? The work W done by gravity? Again, calculate the work done from the definition $W = Fd$. **Key 20**

3. Verify that the ground observer gets the same work-energy equation as the elevator observer. (The work-energy equation means $W = \Delta K$ expressed in terms of m, v, t, g, and h.) **Key 44**

HELPING QUESTIONS

1. What is the length of a unit vector? What is the angle between a unit vector and itself? Between two different unit vectors that point along different coordinate axes? Now you know enough to use the formula for the scalar product. **Key 39**

2. How can you use your results from part (2)? **Key 35**

3. The equation $y = h - gt^2/2$ is valid in the coordinate system that has the positive direction of the y-axis pointing upward. In this coordinate system what is the force $\mathbf{F}$? What is the displacement $\mathbf{d}$ in time t? **Key 45**

4. Draw a force (free-body) diagram to determine the tension in the rope. **Key 43**

5. How is the rope tension related to force F? **Key 33**

6. This is a geometry question, not a physics question. The block finishes h ft closer to the ceiling; how much rope must be pulled past the top pulley on the left? **Key 28**

7. How can you use what you know about circular motion to get the kinetic energy of the mass? (Get rid of v^2!) **Key 37**

8. The only thing you know about the distance traveled is the frictional energy dissipated as the mass slows down and stops. How does this help? **Key 34**

9. What is the total energy of the system when the friction comes on? **Key 16**

10. How does the change in the sum of the kinetic energy plus the potential energy relate to the mechanical energy lost to friction? **Key 4**

11. What is the force of friction? Then what is the mechanical energy lost to friction? **Key 22**

12. What quantity is conserved throughout the motion? **Key 2**

13. What are the initial kinetic and potential energies, K_{i} and U_{i}? What are the final kinetic and potential energies K_{f} and U_{f}? Take the potential energy equal to zero at the lowest point of the motion. **Key 27**

14. Use conservation of energy to determine the speed v of the mass at angle θ. What is the normal force when the block just leaves the sphere? Your answer will give you a second equation relating v and θ. **Key 29**

15. How many stairs can you climb in half an hour? **Key 42**

16. Is the *total* energy of the system conserved? When is *kinetic* energy the greatest? **Key 24**

17. When the potential energy of the system is decreasing, is the sun doing positive or negative work on the planet? **Key 21**

18. What is the sign of $\mathbf{F} \cdot \mathbf{v}$ if the angle between $\mathbf{F}$ and $\mathbf{v}$ is less than $90°$? Greater than $90°$? **Key 13**

Notes: Conserved Quantities

Now that you've finished Learning Guides 2 and 3, you're able to solve mechanics problems in two quite different ways — by $\mathbf{F} = m\mathbf{a}$ and by energy conservation. As you progress in your study of mechanics, you'll develop an intuition for what types of problems are best solved by $\mathbf{F} = m\mathbf{a}$ and what types of problems should be solved by energy conservation. But while you're learning, a good rule to work by is: *always try to solve the problem first by energy conservation.* If the problem can be solved by both $\mathbf{F} = m\mathbf{a}$ and energy conservation, *it's almost always easier to solve it by energy conservation.* Let's look back at some examples.

Problem III, where a mass attached to a spring slows down by friction, illustrates the point well. It's easy to solve this problem by energy methods once you get the hang of it. In principle, it's also possible to solve it using $\mathbf{F} = m\mathbf{a}$; but to do this, you would first need to know an equation describing the path of the mass. Thus, $\mathbf{F} = m\mathbf{a}$ is the hard way to solve the problem. Problem IV, in which a block slides down a ramp, is a less clear-cut case. The helping questions suggest a combination of attacks, using the work-energy theorem together with $\mathbf{F} = m\mathbf{a}$ to get the mechanical energy lost to friction; this, however, is not really much easier than using only $\mathbf{F} = m\mathbf{a}$. Problem V, with the pendulum and the peg, is similar to problem III: solvable by conservation of energy, but practically impossible by $\mathbf{F} = m\mathbf{a}$. Problem VI, where a block is sliding off a sphere, needs a combination of attacks: one equation from from $\mathbf{F} = m\mathbf{a}$ and one from energy conservation.

Soon you will learn about conservation of linear momentum, as well as conservation of angular momentum. Since linear and angular momenta are both vector quantities with three components each, you will have seven conserved quantities in all – quite a powerful arsenal. *You will be able to solve a wide range of physical problems by conservation laws alone. Some of the physical systems that you will study will be so complex that it won't be possible to identify all the forces; but you will still be able to get important results by using conservation laws alone.*

ANSWER KEY

1.
$$\frac{\text{PE (spring)}}{\text{KE (Mass)}} = \frac{R - l_0}{R}$$

2. Total energy

3. Bottom half; Top half

4. The change in mechanical energy and the mechanical energy lost to friction are equal. Both quantities are negative for this and all problems involving friction.

5. $d = 2nh$

6.

$\bullet$	**i**	**j**	**k**
i	1	0	0
j	0	1	0
k	0	0	1

7.
$$W = \Delta K, \quad \frac{mg^2t^2}{2} = \frac{mg^2t^2}{2} - 0$$

8. No, since $Fd = Wh$ for all choices of n. We know from the work-energy theorem that it must be this way.

9. Around 1/6 horsepower, if your mass is 60 kg.

10. Point A

11. $K_0 = 0;\ K = \frac{1}{2}m(gt)^2;\ W = mgh.$

12.
$$\alpha = \cos^{-1}\left[\left(\frac{1}{l-L}\right)\left(l\cos\theta_0 - L - \frac{v_0^2}{2g}\right)\right]$$

13. $+,\ -$

14. Larger

15. It checks: when the planet is going from B to A the angles between $\mathbf{F}$ and $\mathbf{v}$ is *less* than 90°, so the power delivered is positive.

16.
$$E_{\text{tot}} = \frac{kR}{2}(R - l_0) + \frac{1}{2}k(R - l_0)^2$$

17. $mg^2t^2/2$

18. $a_xb_x + a_yb_y + a_zb_z$

19.
$$\alpha = \cos^{-1}\left(\frac{l\cos\theta_0 - L}{l - L}\right)$$

20. $K_0 = \frac{1}{2}mv^2;\quad K = \frac{1}{2}m(-v + gt)^2$ $W = mg(h - vt)$

21. Positive

22.
$$F_k = \mu_k N = \mu_k mg\cos\theta$$
$$W_k = \mathbf{F}_k \cdot \mathbf{d}_k = -\mu_k mgh(\cos\theta/\sin\theta)$$

23. $F = W/2n$

24. Yes; when the potential energy is the least

25. 24

26. No. l_0 cannot be zero for a real spring.

27.
$$K_{\text{i}} = 0;\ U_{\text{i}} = mgl(1 - \cos\theta_0)$$
$$K_{\text{f}} = 0;\ U_{\text{f}} = mg(l - L)(1 - \cos\alpha)$$

28. $2nh$

29. From energy conservation:

$$\frac{1}{2}mv^2 = mgr(1 - \cos\theta).$$

From the normal force equaling zero at breakaway:

$$mg\cos\theta = \frac{mv^2}{r}.$$

30.

$$\text{Distance} = \frac{k}{\mu mg}(R - l_0)\left(R - \frac{l_0}{2}\right)$$

31. $K_0 = 0; \quad K = mg^2t^2/2$

32. 24 again

33. They are equal.

34. Energy lost to friction $= \mu mgd =$ initial total energy.

35. Multiply out $(a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) \cdot (b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k})$. You will get nine terms, and for each one of them you can use an entry from your multiplication table.

36. $\theta = \cos^{-1}(2/3) \simeq 48.2°$, independent of r and g.

37.

$$\text{Force} = k(R - l_0) = \frac{mv^2}{R},$$

so

$$\frac{1}{2}mv^2 = \frac{1}{2}kR(R - l_0).$$

38.

$$\frac{\mathrm{d}W}{\mathrm{d}t} = mg^2t,$$

which agrees with

$$\mathbf{F} \cdot \mathbf{v} = (-mg\mathbf{j}) \cdot (-gt\mathbf{j}) = mg^2t.$$

39. $1; 0°; 90°$; so $\mathbf{i} \cdot \mathbf{i} = (1)(1)\cos 0° = 1$, etc.

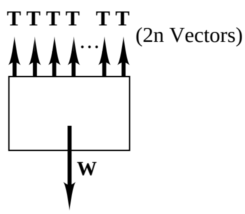
40.

$$v = \sqrt{2gh\,(1 - \mu_k \cot\theta)} = 2.5 \text{ m/s}$$

41. The answer doesn't change.

42. Maybe 1 step per second, or 1800 steps. 5 steps is about 1 m. So you can lift your weight through 360 m in 1800 s. What's your power output?

43.



44. The ground observer gets an extra term, minus $mgvt$, on each side of his equation.

45. $-mg$ and $-gt^2/2$
(note both minus signs)

46. $\theta_{\min} = \tan^{-1}\mu = 5.7°$

Princeton University
Physics 103
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Learning Guide No. 4: Momentum and Collisions

Suggested Reading: Tipler Chapter 8 (Optional: Section 8-8)

Problem I

Find the centers of mass of the following collections of objects.

1. Two masses of 3 kg and 5 kg, separated by a distance of 10 m. **Key 10**
2. The following collection of masses:

   2 kg at $(0, 0, 0)$ m

   6 kg at $(0, 0, 4)$ m

   6 kg at $(0, 4, 0)$ m

   6 kg at $(4, 0, 0)$ m

   3 kg at $(1, 1, 1)$ m

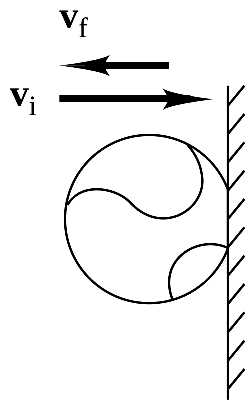
   5 kg at $(-1, 0, 2)$ m

   2 kg at $(-3, 2, -7)$ m **Key 13**
3. A solid sphere of constant density **Key 30**
4. A thin spherical shell of constant density and thickness **Key 18**
5. A thin Hula-Hoop of constant density and thickness **Key 3**

Problem II

A 198-lb man is carrying three coconuts each weighing one pound. He wants to cross a bridge that will hold 200 lb but not one ounce more. He decides to juggle the balls while crossing the bridge so that one ball will always be in the air. Does he make it across the bridge? If this doesn't seem like physics, look at Helping Questions 1 and 2. **Key 14**

Problem III

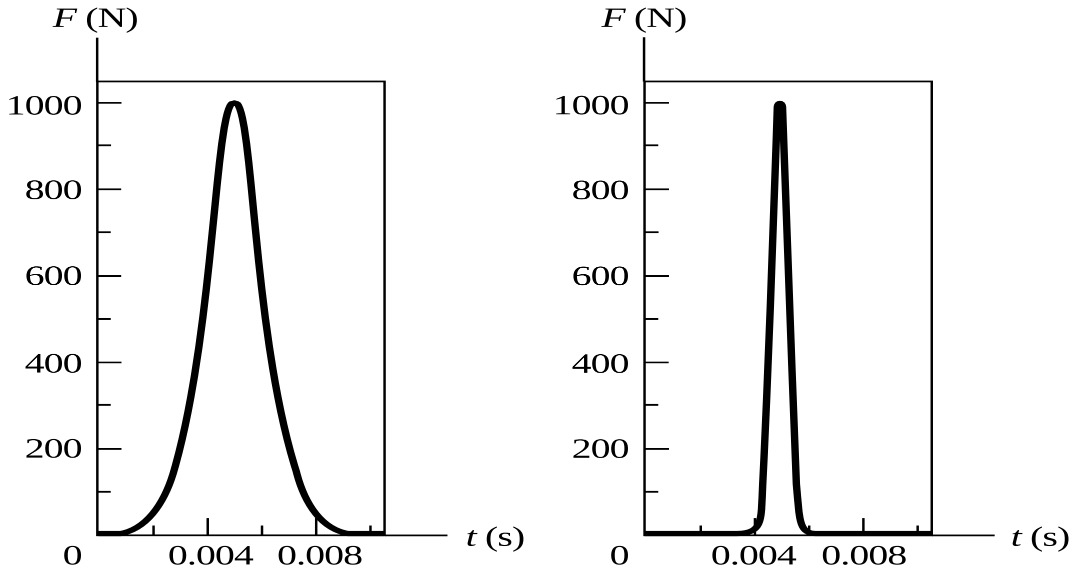


A tennis ball of mass $m = 50$ g is hit at 30 m/s against a backboard. The ball returns with a speed of 20 m/s. In your analysis, neglect the vertical motion of the ball.

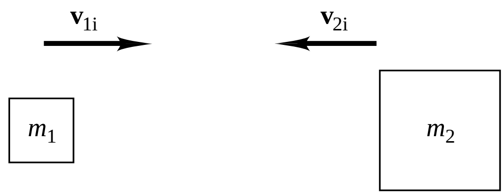
1. What is the change Δp in the ball's momentum? **Key 29**

2. What is the impulse J given to the ball? **Key 11**

3. Which of the graphs below could be a graph of the force on the backboard versus time? **Key 24**



Problem IV



A block with mass $m_1 = 10$ kg moving at 5 m/s collides with another block with mass $m_2 = 20$ kg moving the other way at 1 m/s. The two blocks stick together after the collision.

1. What is their common final velocity v_f? Refer to Helping Question 3 if you get stuck. **Key 33**

The blocks collide again, this time elastically.

2. What are the final velocities $\mathbf{v}_{1\mathrm{f}}$ and $\mathbf{v}_{2\mathrm{f}}$? Assume that the outgoing blocks move away from the collision along the initial line of approach. If you need help somewhere along the line, use Helping Questions 4 and 5. **Key 34**

Problem V

A Volkswagen weighing 1600 lb and a Mercedes weighing 4000 lb, each moving at 44 ft/s (30 mi/h), enter an intersection from the north and south, respectively. They collide head on, and the resulting junk sticks together.

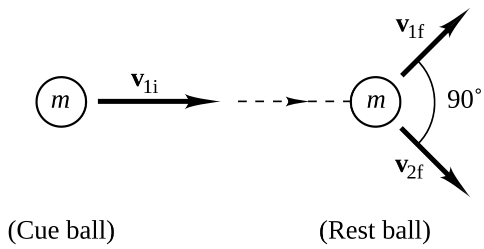
1. What is the final velocity of the junk? If you don't understand, refer to Helping Question 6. **Key 9**
2. What is the change in velocity experienced by the passenger in each car? **Key 35**

For the purpose of estimation, assume that the collision lasts 0.1 s and that the deceleration of the cars is constant.

3. What is the magnitude of the deceleration of each car in terms of g, the gravitational constant? Take $g = 32$ ft/s^2. **Key 26**
4. If the Mercedes enters the intersection from the east, instead of from the south, what is the magnitude and direction of the junk's velocity? **Key 22**

Problem VI

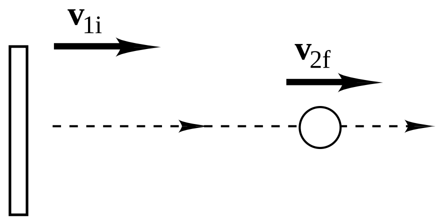
In an elastic collision between two particles of *equal* mass, one of which is initially at rest, the recoiling particles always move off at right angles to one another. This may be proved directly from the equations of conservation of energy and momentum.



1. Write the *vector* equation for conservation of momentum. What does this imply about the relation between the velocities? **Key 21**
2. Write the equation for conservation of energy. What does this imply about the relationship between the speeds? **Key 2**
3. Take the scalar (dot) product of the relation between the velocities from part (1) with itself. **Key 5**
4. Compare the results of parts (2) and (3). What does this tell you about $\mathbf{v}_{1f} \cdot \mathbf{v}_{2f}$? What do you conclude? **Key 31**

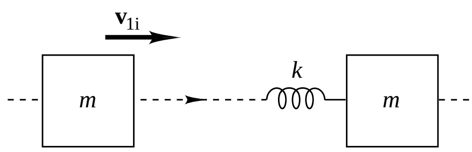
Problem VII

An "infinitely massive" Ping-Pong paddle moving with velocity $\mathbf{v}_{1\mathrm{i}}$ hits a "massless" Ping-Pong ball at rest. The collision is elastic. What is the final velocity of the ball, $\mathbf{v}_{2\mathrm{f}}$? If you feel you have to resort to lengthy computations to get the answer, see Helping Questions 7 and 8 for a quicker way. **Key 6**



Problem VIII

A block of mass m moving with velocity $\mathbf{v}_{1\mathrm{i}}$ collides elastically with a stationary block, also of mass m. The massless spring of spring constant k is compressed for a short time during the collision. What is the maximum compression x of the spring? If you're lost, follow Helping Questions 9, 10, and 11. **Key 19**



Problem IX

A billiard ball moving at a speed of 2.2 m/s strikes an identical stationary ball a glancing blow. After the collision, one ball is found to be moving at a speed of 1.1 m/s, in a direction making a 60° angle with the original line of motion.

1. What are the magnitude and direction of the other ball's velocity? **Key 8**
2. Is the collision inelastic? **Key 1**

Problem X

A fisherwoman in a canoe catches a very big fish. She observes that as she reels in 100 m of line to bring the fish into her lap at the center of the canoe, she (mass 70 kg) and the canoe (mass 25 kg) move 10 m toward the original position of the now quiescent fish, and away from a buoy at their original position. How heavy is the fish she caught? (No exaggeration, please!). Assume that the fish and the boat are at the same height (i.e., both at the surface); furthermore, ignore the frictional interaction with the water and the size of the canoe. Stuck? Check out Helping Question 12. **Key 20**

HELPING QUESTIONS

1. Describe in words the motion of the center of mass of the man-coconuts system as it moves across the bridge. **Key 7**

2. What is Newton's second law for a *system* of particles? Can you use this to find the normal force acting upward of the man's feet? How does this relate to the force of the man's feet acting downward on the bridge? **Key 17**

3. What quantity is conserved? What equation relating the masses and velocities does this conservation law give? **Key 28**

4. What quantities are conserved? What equations relating the masses and the velocities do the conservation laws give? **Key 25**

5. Solving the two equations given by the conservation laws for $v_{1\mathrm{f}}$ and $v_{2\mathrm{f}}$ in terms of m_1, $v_{1\mathrm{i}}$, m_2, and $v_{2\mathrm{i}}$ is a little tricky. If you don't see how to start, try collecting terms containing m_1 on one side of the equations and terms containin m_2 on the other side of the equations. Can you see what to do now? If not, try dividing the energy equation by the momentum equation. Now what do you have? You should have a linear equation in v_{1i}, v_{2i}, v_{1f}, and v_{2f}. Your original conservation of momentum equation is also a linear equation in v_{1f} and v_{2f}, so you have two linear equations in two unknowns, which you should be able to solve. What do you get for v_{1f} and v_{2f}? **Key 36**

6. What quantity is conserved? What equation relating the masses of the two cars and the initial and final velocities does this equation give? **Key 4**

7. How would an observer moving from left to right at speed $v_{1\mathrm{i}}$ with the paddle describe the collision? At what speeds would he see the ball, before and after the collision? **Key 15**

8. Since the observer moving with the paddle sees the ball go off to the right at speed $v_{1\mathrm{i}}$, what ball speed does an observer in the original frame see? **Key 23**

9. Think about the instant of maximum compression. What is the *relative velocity* of the two blocks at that instant? What is the velocity of each block at the instant of maximum compression? **Key 16**

10. What is the total energy of the system at the instant of maximum compression in terms of the variables given in the problem? **Key 12**

11. What is the initial kinetic energy of this system? **Key 32**

12. Where is the center of mass of the woman, rowboat, and fish after the fish has been reeled in? **Key 27**

Notes: Number of Unknowns and Number of Equations

As you do the problems in these Learning Guides, you may sometimes get the feeling that the whole business is getting out of hand — there are so many variables that you can't possibly keep track of all of them.

An excellent way to organize your thoughts is to divide the variables into two groups, knowns and unknowns. A known variable doesn't necessarily have to be known in the sense that you know its numerical value. A known variable is a variable you're allowed to leave in the answer. An unknown variable is a variable you want to express in terms of the known variables. The reason that this distinction is so useful is that *you need to find as many different equations as there are unknowns.* This is only a rule of thumb, not a theorem, but it works very well. Let's consider, as examples, three of the problems in this Learning Guide:

Problem		Knowns	Unknowns	Equations
IV, V	Inelastic collisions	$m_1, v_{1\text{i}}, m_2, v_{2\text{i}}$	v_f	Conservation of momentum in the x or y direction
IV	1-D elastic collisions	$m_1, v_{1\text{i}}, m_2, v_{2\text{i}}$	$v_{1\text{f}}, v_{2\text{f}}$	Conservation of momentum in the x direction; conservation of energy
VI, IX	2-D elastic collisions	$m_1, v_{1\text{i}}, m_2, v_{2\text{i}}$	$v_{1\text{f}}, v_{2\text{f}}, \theta$	Conservation of momentum in the x and y directions (2 equations); conservation of energy

Suppose you wanted to solve the two-dimensional elastic collision completely — that is, you wanted to find not only the angle θ between the paths of the recoiling particles, but also the angles θ_1 and θ_2 that the paths make with the x-axis. Then you would have four unknowns $v_{1\text{f}}$, $v_{2\text{f}}$, θ_1, and θ_2, and no matter how you floundered about using the three conserved quantities p_x, p_y, and E, you could not solve the problem. This has a physical interpretation: the recoiling velocity vectors are not determined by p_x, p_y, and E alone; they depend on the details of the forces in the collisions.

You might want to examine a few of your solutions to problems in previous Learning Guides to verify that the number of unknowns is equal to the number of equations. *Note that for a given problem the number of unknowns is not fixed.* You might introduce an unknown frictional force F_k while solving a force problem, while someone else might immediately write down $\mu_k N$. You would have an extra unknown but you would also have an extra equation $F_k = \mu_k N$. Another good point about this way of organizing your thoughts is that you can see right away where the physics ends and the algebra begins: *the physics ends when you have found enough equations to determine your unknowns.*

Choice of Reference Frame

Turning to another subject, Newton's laws (and thus the conservation laws you have learned) are valid in any *inertial* reference frame. But for a given situation, as you saw in Problem VII, certain reference frames may be much more useful than others. What reference frame was used in Problem VII? *The center-of-mass frame.* What's so special about the center-of-mass frame? (See Sections 8-1 to 8-3 in Tipler for the definition of center of mass and center-of-mass velocity.)

First, let's look at what happens to the kinetic energy

$$K = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

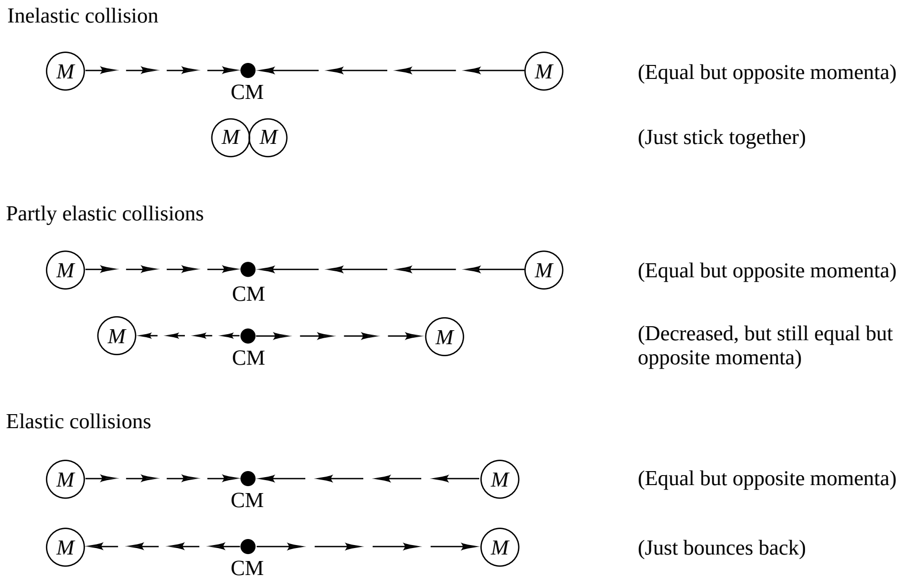
in a one-dimensional collision. The kinetic energy can be divided into two types: kinetic energy *of* the center of mass, here called K_{CM}, and kinetic energy *with respect to* the center of mass, here called K_{internal}. Let $\mathbf{V}$ represent the velocity of the center of mass. Then the kinetic energy of the center of mass is

$$K_{\mathrm{CM}} = \frac{1}{2}(m_1 + m_2)\mathbf{V}^2$$

and the kinetic energy with respect to the center of mass is

$$K_{\mathrm{internal}} = \frac{1}{2}m_1 \left(\mathbf{v}_1 - \mathbf{V}\right)^2 + \frac{1}{2}m_2 \left(\mathbf{v}_2 - \mathbf{V}\right)^2$$

Nothing happens to K_{CM} in a collision, since $\mathbf{V}$ remains constant. K_{internal} always decreases as the particles collide and then returns to some fraction of its initial value depending on how elastic the collision is. If you didn't use the center-of-mass frame to solve Problem VIII, go back and write the energy equations as viewed from a frame moving to the right with speed $v_{1i}/2$ $(= \mathbf{V})$. Notice that in this frame it is easier to get the spring compression x because all and not just some of the kinetic energy is momentarily transferred to potential energy of the spring. All types of collisions look simpler when you move along with the center of mass in such a way that the center of mass is fixed in your frame:



Note that the way the figure is drawn, the velocity vectors are not the same; so the masses must be different!

The Helping Questions helped you through the algebra of solving problem IV. See if you can do it a simpler way using the center-of-mass frame!

If you keep these two ideas in mind — number of equations versus number of unknowns and choice of reference frame — the problems in the next Learning Guide will be quite a bit easier.

ANSWER KEY

1. No
2. $$\frac{1}{2}mv_{1\mathrm{i}}^2 = \frac{1}{2}mv_{1\mathrm{f}}^2 + \frac{1}{2}mv_{2\mathrm{f}}^2,$$
$$\text{so } v_{1\mathrm{i}}^2 = v_{1\mathrm{f}}^2 + v_{2\mathrm{f}}^2.$$
3. At the center
4. Linear momentum is conserved, so
$$m_1 v_{1\mathrm{i}} + m_2 v_{2\mathrm{i}} = (m_1 + m_2)v_\mathrm{f}$$
5. $$v_{1\mathrm{i}}^2 = v_{1\mathrm{f}}^2 + 2\mathbf{v}_{1\mathrm{f}} \cdot \mathbf{v}_{2\mathrm{f}} + v_{2\mathrm{f}}^2$$
6. $v_{2\mathrm{f}} = 2v_{1\mathrm{i}}$
7. The center of mass moves almost horizontally across the bridge at whatever speed the man is walking. It bobs up and down a little, because of the juggling of the coconuts and the "bounce" in the man's step. However, there certainly is some time when the vertical acceleration of the center of mass is zero.
8. 1.9 m/s, 30° from $\mathbf{v}_\mathrm{i}$.
9. $$v = \frac{W_1 v_{1\mathrm{i}} + W_2 v_{2\mathrm{i}}}{W_1 + W_2} = 12.9 \text{ mi/h} = 18.9 \text{ ft/s}$$
in the direction of the Mercedes.
10. $3\frac{3}{4}$ m from the 5 kg mass on the line segment between the two masses
11. -2.5 kg·m/s ($\mathbf{J}$ points to the left)
12. $$\frac{1}{4}mv_{1\mathrm{i}}^2 + \frac{1}{2}kx^2$$
13. $\left(\frac{8}{15}, \frac{31}{30}, \frac{23}{30}\right)$ m
14. No. It might have collapsed even without the coconuts. See Helping Question 1.
15. An observer would say that a ball moving at speed $v_{1\mathrm{i}}$ to the left hits a solid, stationary wall. So, since the collision is elastic, the ball just bounces back with speed $v_{1\mathrm{i}}$.
16. 0, $v_{1\mathrm{i}}/2$
17. $\mathbf{F}_\mathrm{ext} = M\mathbf{a}_\mathrm{cm}$. Yes, when there is no vertical acceleration the normal force is exactly 201 lb. They are equal.
18. At the center.
19. $$x = \sqrt{\frac{m}{2k}}v_{1\mathrm{i}}$$
20. 10.56 kg
21. $m\mathbf{v}_{1\mathrm{i}} = m\mathbf{v}_{1\mathrm{f}} + m\mathbf{v}_{2\mathrm{f}}$ so, $\mathbf{v}_{1\mathrm{i}} = \mathbf{v}_{1\mathrm{f}} + \mathbf{v}_{2\mathrm{f}}$.
22. 33.8 ft/s; 68.2° west of south
23. $2v_{1\mathrm{i}}$
24. The first
25. Conservation of momentum gives
$$m_1 v_{1\mathrm{i}} + m_2 v_{2\mathrm{i}} = m_1 v_{1\mathrm{f}} + m_2 v_{2\mathrm{f}}.$$
Conservation of energy gives
$$\frac{1}{2}m_1 v_{1\mathrm{i}}^2 + \frac{1}{2}m_2 v_{2\mathrm{i}}^2 = \frac{1}{2}m_1 v_{1\mathrm{f}}^2 + \frac{1}{2}m_2 v_{2\mathrm{f}}^2.$$
26. $a = 7.8g$ for the Mercedes and $a = 19.7g$ for the Volkswagen.
27. At the center of the boat, 10 m from the buoy
28. Linear momentum is conserved, so,
$$m_1 v_{1\mathrm{i}} + m_2 v_{2\mathrm{i}} = (m_1 + m_2)v_\mathrm{f}.$$

29. $m(v_\mathrm{f} - v_\mathrm{i}) = -2.5$ kg·m/s (taking the positive direction to the right).

30. At the center

31. $\mathbf{v}_{1\mathrm{f}} \cdot \mathbf{v}_{2\mathrm{f}} = 0$, so either one of the balls is stopped, or, they're moving at right angles.

32. $mv_{1\mathrm{i}}^2/2$

33.
$$v_\mathrm{f} = \frac{m_1 v_{1\mathrm{i}} + m_2 v_{2\mathrm{i}}}{m_1 + m_2} = 1 \text{ m/s}$$
to the right.

34. $\mathbf{v}_{1\mathrm{f}} = -3$ m/s; $\mathbf{v}_{2\mathrm{f}} = 3$ m/s

35. 25.1 ft/s for the Mercedes passenger and 62.9 ft/s for the Volkswagen passenger

36.
$$v_{1f} = \frac{2m_2 v_{2i} + (m_1 - m_2)v_{1i}}{m_1 + m_2}$$
$$v_{2f} = \frac{2m_1 v_{1i} + (m_2 - m_1)v_{2i}}{m_1 + m_2}$$

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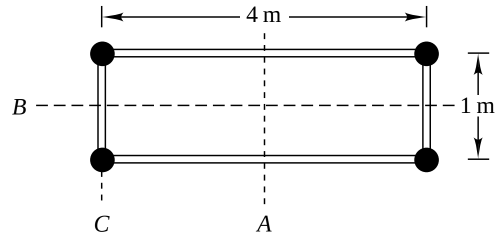
Learning Guide No. 5: Rotation and Rolling

Suggested Reading: Tipler, Chapter 9

Problem I

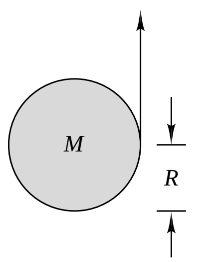


The rigid body shown in the figure consists of four 10 kg spheres connected by four light rods. Treat the spheres as point particles and neglect the mass of the rods.

1. Which is greater, the rotational inertia about axis A or the rotational inertia about axis B? What are their exact values? **Key 11**

2. Use the parallel-axis theorem to calculate the rotational inertia about axis C. Check your answer by calculating the rotational inertia about axis C from the definition. **Key 3**

Problem II

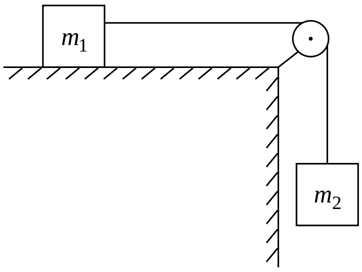


A string wrapped around a solid cylinder of mass M and radius R is pulled vertically upward to prevent the cylinder from falling as it unwinds the string (i.e., the center-of-mass of the cylinder does not move).

1. What is the tension in the string? If you disagree with the key, use Helping Question 1. **Key 14**

2. If the cylinder is initially at rest, how much string is unwound after a time t? If, after a good effort, you're stuck, use Helping Questions 2 and 3. **Key 17**

Problem III



In the apparatus shown in the sketch, both blocks accelerate as a result of the force of gravity on m_2. The coefficient of kinetic friction between m_1 and the table is μ_k. The pulley has frictionless bearings, moment of inertia I, and radius R. The string does not slip on the pulley, so the pulley undergoes an angular acceleration. The blocks are initially at rest. At time t, through what distance y has m_2 moved? Use Helping Questions 4, 5, and 6 if you need to. **Key 9**

Problem IV

After the block of mass m_2 in Problem III has fallen a distance y from rest, it has speed v. Use the work-energy theorem to express v in terms of m_1, m_2, μ_k, g, y, I, and R. Use Helping Questions 7, 8, and 9 if necessary. **Key 15**

Problem V

A pool cue strikes a pool ball which is sitting on a level pool table with a coefficient of kinetic friction μ_k. The ball is given an initial speed of v_0 with no spin. How fast is the ball moving when it begins to roll without slipping? Even though the center-of-mass frame is a *non-inertial reference frame*, $\tau = I\alpha$ still holds in this frame — see Section 9-6 of Tipler. If you need more hints, use Helping Questions 10, 11, and 12. **Key 18**

HELPING QUESTIONS

1. Is the cylinder's center of mass moving? Then what must be the sum of the external forces on the cylinder? **Key 4**
2. What is the angular acceleration of the cylinder? **Key 10**
3. What is the rotational analogue of the translational kinematic formula $\Delta x = v_0 t + \frac{1}{2}at^2$? **Key 12**
4. Draw force (free-body) diagrams for each block and for the pulley. Call the tension in the horizontal part of the string T_1 and the tension in the vertical part of the string T_2. What is the torque τ on the pulley in terms of T_1 and T_2? What are T_1 and T_2 in terms of μ_k, m, g, and a, the acceleration of the blocks? **Key 1**
5. Can you relate the torque on the pulley to its angular acceleration? Can you relate the angular acceleration to the linear acceleration of the blocks? Now can you solve for the acceleration a? **Key 5**
6. What did you get for the acceleration a? What kinematic formula would be appropriate to get the distance y? **Key 7**
7. What is the change in potential energy? **Key 6**
8. What is the change in kinetic energy? **Key 2**
9. How much mechanical energy is lost to friction? **Key 8**
10. What is the condition required for rolling without slipping? **Key 16**
11. What are the forces on the ball? Then what is $v(t)$? **Key 19**
12. What are the torques on the ball? Then what is $\omega(t)$? **Key 13**

Notes: Analogies

You're probably finding rotational motion to be the hardest subject to understand in this course so far. It's no wonder. First, you've been introduced to a whole slew of physical concepts all at once — six, to be exact, $\theta, \omega, \alpha, \boldsymbol{\tau}, \mathbf{L}$, and I. Worse yet, you have to be very precise when you write down equations describing rotational motion. Torques have to be defined with respect to a point (not an axis!) and rotational inertias have to be defined with respect to an axis (not a point!). You have to define your variables very carefully even when you write down an innocent looking equation like $v = \omega r$, since this may hold only under certain conditions.

Is there anything to guide you through this maze of subtleties? Yes — *analogies* with what you've learned previously! You should be able to reproduce Table 9-2 in Tipler without any strain. $\tau = I\alpha$ shouldn't seem like a completely new equation to you — it's just the rotational analogue of $F = ma$. What are the linear analogues of the following equations?

$$\begin{aligned} \theta &= \omega_0 t + \frac{1}{2}\alpha t^2 \\ \mathbf{L} &= I\boldsymbol{\omega} \\ P &= \boldsymbol{\tau} \cdot \boldsymbol{\omega} \\ E &= \frac{1}{2} I \omega^2 \end{aligned}$$

You might get the impression from these tables that the equations you need to solve physics problems fall roughly into two categories — equations among linear variables and equations among rotational variables. There is really also a third category — equations relating linear variables to rotational variables. Every analogy between a rotational variable and a linear variable (except for the analogy between θ and x) corresponds to an equation relating the two variables and r. For kinematic variables, the r is on the rotational variable's side:

$$\begin{aligned} v &= \omega r \\ a &= \alpha r, \end{aligned}$$

where a means **tangential** acceleration. For dynamical variables, the r is on the linear variable's side:

$$\begin{aligned} \boldsymbol{\tau} &= \mathbf{r} \times \mathbf{F} \\ \mathbf{L} &= \mathbf{r} \times \mathbf{p} \end{aligned}$$

All equations work out dimensionally because there are two r's involved in the analogy between mass and rotational inertia:

$$I = (\text{constant}) m r^2.$$

Of course, you can't just write down a bunch of equations "by analogy" and expect to solve a physics problem: you have to examine each problem separately and carefully define your variables so that your equations make sense. But the analogies serve as a useful guide. As an example, let's look at the solution suggested by the Helping Questions to Problem III.

$$\begin{aligned}
T_1 - \mu_k m_1 g &= m_1 a && \text{(linear)} \\
m_2 g - T_2 &= m_2 a && \text{(linear)} \\
\tau &= R(T_2 - T_1) && \text{(both)} \\
\tau &= I\alpha && \text{(rotational)} \\
a &= R\alpha && \text{(both)} \\
y &= at^2/2 && \text{(linear)}
\end{aligned}$$

(six unknowns: T_1, T_2, a, α, t, and y).

ANSWER KEY

1. $\tau = R(T_2 - T_1)$ – into the page;

   $m_2 g - T_2 = m_2 a$;

   $T_1 - m_1 \mu_k g = m_1 a$.

2. $$K_\mathrm{f} - K_\mathrm{i} = \frac{1}{2} m_1 v^2 + \frac{1}{2} I\omega^2 + \frac{1}{2} m_2 v^2$$

3. $I_C = 320$ kg·m^2

4. No; 0

5. $\tau = I\alpha$;   yes : $a = R\alpha$

6. $U_\mathrm{f} - U_\mathrm{i} = -m_2 g y$

7. $$a = \frac{m_2 g - m_1 \mu_k g}{m_1 + m_2 + I/R^2};$$
   $$y = v_{0y} t + \frac{1}{2} a t^2$$

8. $\mu_k m_1 g y$

9. $$y = \frac{1}{2}\left(\frac{m_2 g - m_1 \mu_k g}{m_1 + m_2 + I/R^2}\right) t^2$$

10. $2g/R$

11. $I_A > I_B$; $I_A = 160$ kg·m^2 and $I_B = 10$ kg-m^2

12. $\Delta\theta = \omega_0 t + \alpha t^2/2$

13. $\mu_k m g r$; $\omega(t) = \mu_k m g r t / I$

14. Mg

15. $$v = \sqrt{\frac{(m_2 - m_1 \mu_k)}{(m_1 + I/R^2 + m_2)} 2gy}$$

16. $v = \omega r$

17. $g t^2$

18. $$v = \frac{v_0}{(1 + I/mr^2)} = \frac{5}{7} v_0$$

19. $-\mu_k m g$; $v(t) = v_0 - \mu_k g t$

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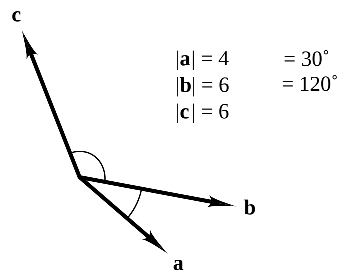
Learning Guide No. 6: Angular Momentum

Suggested Reading: Tipler, Chapter 10 (Optional: Section 10-4)

Vector Warm-Ups: The Vector Product

See Section 10-1 of Tipler to learn about vector products (also known as "cross products").

The vectors **a**, **b**, and **c** all lie in the plane of the paper. Give the magnitude and direction of the following vector products:

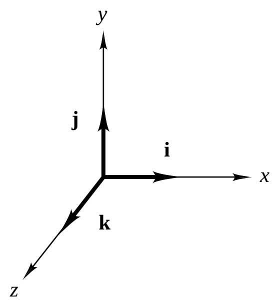


1. $\mathbf{a} \times \mathbf{b}$. **Key 18**
2. $\mathbf{a} \times \mathbf{c}$ **Key 19**
3. $\mathbf{a} \times \mathbf{a}$ **Key 25**
4. $\mathbf{b} \times \mathbf{a}$ **Key 24**

There is another method that is occasionally more convenient for calculating vector products. You will discover the method in parts (5) and (6) and then use it in part (7).

5. **i**, **j**, and **k** are the unit vectors drawn in the diagram. Fill in the multiplication table below:

$\times$	**i**	**j**	**k**
i			
j			
k			

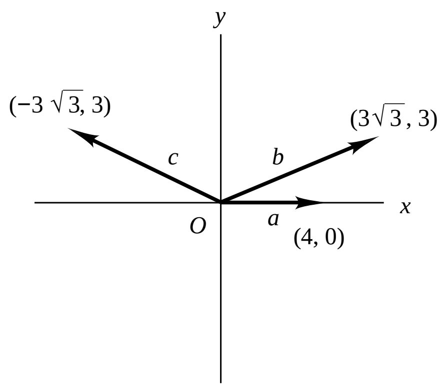


If you need help, turn to Helping Question 1. **Key 21**

6. In a certain fixed coordinate system,

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \quad \text{and} \quad \mathbf{b} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}.$$

What is $\mathbf{a}\times\mathbf{b}$ in terms of the unit vectors and the components of $\mathbf{a}$ and $\mathbf{b}$? If both $a_z = 0$ and $b_z = 0$, what does your answer reduce to? See Helping Question 2. **Key 23**



7. The vectors from parts (1) through (4) are given in the adjacent sketch in terms of their components in a rectangular coordinate system. Check your answers to parts (1) to (4) using the new method.

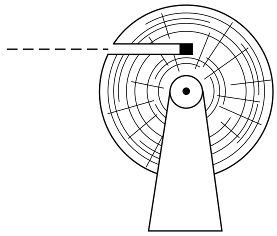
Problem I

A diver does a two-and-a-half front somersault from a 3-meter board. While she is in her tuck, the rotational inertia of her body is 4 slugs $\times$ (1 ft)^2. She is spinning at a rate of one complete revolution per second. After she extends her body to enter the water, her rotational inertia is 4 slugs $\times$ (2 ft)^2.

1. How long would it take her to complete one revolution at her new rate of rotation? If you're confused, try Helping Question 3. **Key 14**

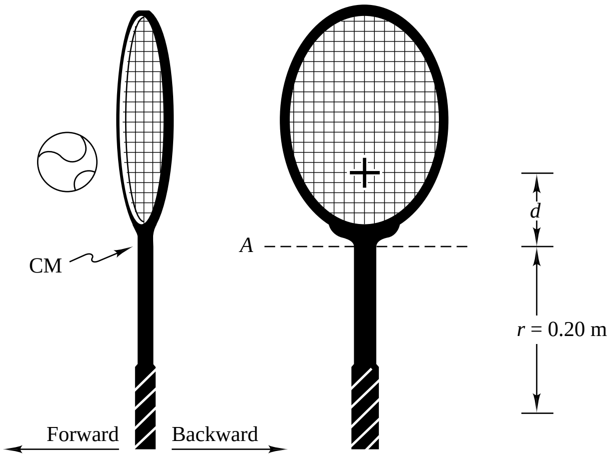
2. Did her rotational kinetic energy increase or decrease? How do you account for the change? If you need a hint for the first question, use Helping Question 4. **Key 2**

Problem II



A disk of wood is mounted on frictionless bearings, leaving it free to rotate about its center. The mass of the disk is 1 kg and its radius is 10 cm. A 5-g bullet traveling at 500 m/s lodges in the disk 5 cm above its center, as indicated on the diagram. The disk starts to rotate. Assume that the displacement of the wood is negligible, and ignore the rotational inertia due to the bullet. How many seconds does it take for the disk to make one revolution after the bullet strikes? See Helping Questions 5 and 6. **Key 10**

Problem III



The "sweet spot" of a tennis racket is the spot where the player feels the least vibration through his wrist and forearm when he hits the ball. In this problem you will ignore many factors but still make an accurate computation of the location of the sweet spot. Assume that there are no external forces on the racket shown in the sketch. The sweet spot of the racket is indicated by a cross. When the ball hits the stationary racket on the sweet spot, the velocity of the center of the grip right after the collision is zero (so that if a player were holding it, he wouldn't feel the grip jar forward or backward during the collision). The mass of the tennis racket is $m = 0.5$ kg, and its moment of inertia about axis A through the center of mass is $I_A =$(0.5 kg)(0.15 m)^2.

1. If the ball hits the racket *beneath* the sweet spot, will the initial velocity of the grip be forward or backward? Use Helping Question 7 if you don't know. **Key 16**

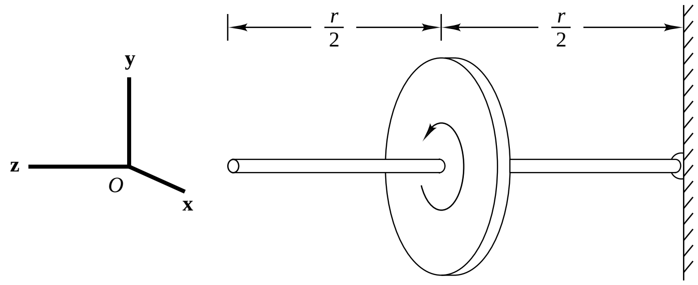
2. If the ball hits the racket *above* the sweet spot, will the initial velocity of the grip be forward or backward? **Key 1**

3. What is the distance d between the center of mass and the sweet spot? Stuck? Use Helping Questions 8 and 9. **Key 4**

Problem IV

In Problems I through VIII, you did not get to use the *vector* equation $\boldsymbol{\tau} = d\mathbf{L}/dt$. In part (2) of this problem you will!

A heavy wheel rotates with no friction and a very high angular velocity $\boldsymbol{\omega}$ about a massless shaft. One end of the shaft is fixed to the wall, but the shaft is free to pivot about this point. The other end of the shaft is held by a physics student. The wheel has mass M and rotational inertia I about the shaft.



1. The student wants to hold the shaft fixed. What force must she apply to the end of the shaft? **Key 7**
2. The student now wishes to raise the shaft by imparting a velocity $\mathbf{v} = v\mathbf{j}$ to the end of the shaft. What force should she apply to the end of the shaft? Be sure to specify the direction of the force! You may assume that since $\boldsymbol{\omega}$ is so large, the angular momentum $\mathbf{L}$ about the pivot always points along the direction of the shaft. For a hint, look at Helping Question 10. **Key 5**

HELPING QUESTIONS

1. What is the length of a unit vector? What is the angle between a unit vector and itself? Two different vectors? How can you figure out the *direction* of a cross product? **Key 20**

2. How can you use your multiplication table? **Key 22**

3. What physical quantity is conserved throughout the dive? Since I increases by a factor of four, what happens to ω? **Key 15**

4. What is the formula for rotational kinetic energy in terms of I and ω? **Key 8**

5. What physical quantity is conserved throughout the collision? Why? **Key 3**

6. What was the angular momentum about the center of the wheel before the collision? **Key 6**

7. Consider an extreme case: what would happen if the ball hit the racket at the level of the center of mass? **Key 12**

8. The racket recoils with velocity v_{CM} and angular velocity ω about its center of mass. What relation must hold between v and ω for the center of the grip to have zero velocity immediately after impact? **Key 17**

9. If the linear momentum the ball transfers to the racket is p, what is the racket's angular momentum about its center of mass? **Key 13**

10. What is $\mathrm{d}\mathbf{L}/\mathrm{dt}$ in terms of I, ω, r, and $\mathbf{v}$? **Key 9**

ANSWER KEY

1. Forward
2. The kinetic energy decreases; the diver does *negative internal work* as she lets her arms and legs "fly out."
3. The *total angular momentum* of the bullet plus the disk about the center of the disk is conserved because there are no external torques about the center of the disk.
4. $$d = \frac{I}{mr} \cong 11 \text{ cm}$$
5. $$\frac{I\omega v}{r^2}\mathbf{i} + \frac{mg}{2}\mathbf{j},$$
   plus an arbitrary force along the z-axis
6. $L = pr_\perp = 0.125$ kg·m^2/s, where $r_\perp$ is the distance of closest approach between the bullet's trajectory and the center of the wheel.
7. $mg/2$ (up) $= mg/2\mathbf{j}$ (plus an arbitrary force in the z-direction)
8. $I\omega^2/2$
9. $I\omega\mathbf{v}/r$
10. About 0.25 s
11. $2g/R$
12. The racket would move backward without spinning.
13. pd
14. 4 s
15. The diver's angular momentum about her center of mass; ω decreases by a factor of 4.
16. Backward
17. $v_{\mathrm{CM}} = \omega r$
18. 12, out of the page
19. 12, out of the page
20. 1; 0; 90°; by the right hand rule
21.

$\times$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	0	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$-\mathbf{k}$	0	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	0

22. Multiply out
   $$\mathbf{a}\times\mathbf{b} = (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) \times (b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}).$$
   You'll get nine terms. For each one you can use an entry from your multiplication table.
23. $$\begin{aligned}\mathbf{a}\times\mathbf{b} = &(a_y b_z - a_z b_y)\mathbf{i}\\ &+(a_z b_x - a_x b_z)\mathbf{j}\\ &+(a_x b_y - a_y b_x)\mathbf{k},\end{aligned}$$
   which reduces to
   $$\mathbf{a}\times\mathbf{b} = (a_x b_y - a_y b_x)\mathbf{k}.$$
24. 12, into the page
25. 0

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Learning Guide No. 7: Gravity

Suggested Reading: Tipler, Chapter 11

Problem I

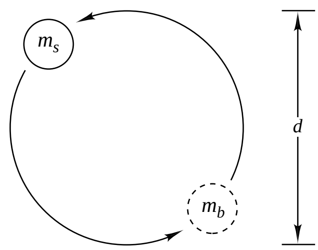
1. The moon circles the earth with period $T = 27.3$ days. The distance from the center of the earth to the center of the moon is $R = 3.85 \times 10^5$ km. What is G, the gravitational constant, in terms of T, R, and M, the mass of the earth? If you're having trouble, use Helping Questions 1 and 2. **Key 32**
2. What is G in terms of M only? Use SI units. **Key 28**
3. An apple falls to the earth with acceleration $g = 9.8$ m/s^2. The radius of the earth is $r = 6370$ km. What is G in terms of g, r, and M? For assistance, see Helping Question 3. **Key 15**
4. What is G in terms of M only? Use SI units. **Key 26**

Problem II

Is it possible for an orbiting satellite always to remain above the same point on the earth? If so, how many kilometers above the earth's surface must it be? Use the value for GM that you calculated in Problem I. Look at Helping Questions 4 and 5 if you need them. **Key 12**

Problem III

A distant star, observed by telescope over the course of many years, is found not to remain fixed but rather to execute uniform circular motion for no apparent reason. It is hypothesized that the cause of the star's motion is a *black hole*. One possibility is sketched in the diagram. The star and the black hole move on the same circular path of diameter d.

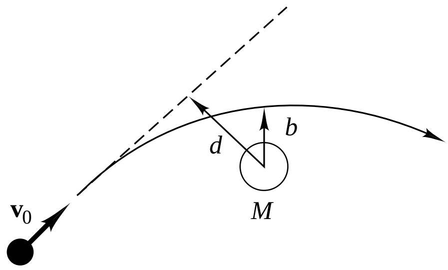


1. What can you say about the mass of the star m_s and the mass of the black hole m_b? Can't see any relation? Look at Helping Question 6. **Key 24**

2. The period of the star's motion is T. What is the mass of the black hole in terms of G, T, and d? If need be, see Helping Questions 7 and 8. **Key 13**

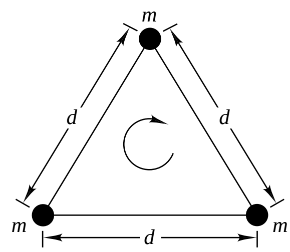
Problem IV

A meteor moves toward the solar system with speed v_0 in a direction such that it would miss the sun by a distance d if it were not attracted by the sun's gravitational force. Denote the mass of the sun by M. Find the distance b of the meteor from the sun at the point of closest approach in terms of v_0, d, and M (and G, the gravitational constant). Turn to Helping Questions 9 to 11 if you're stumped. **Key 9**



Problem V

Three particles, of mass m each, are located at the vertices of an equilateral triangle and are spinning about their center of mass in space that is otherwise empty. The sides of the triangle are of length d, which does not change with time.

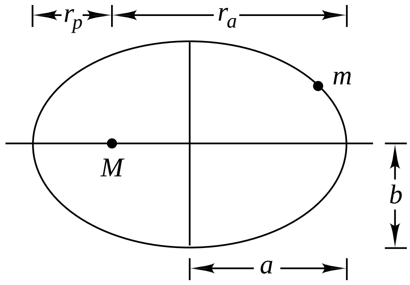


1. What is the potential energy of the system? If you don't see how to get the answer, reread Section 11-3 of Tipler. **Key 2**

2. What is the kinetic energy of the system in terms of G, m, and d? Puzzled? See Helping Questions 12, 13, and 14. **Key 4**

Problem VI

Tipler verifies Kepler's third law for circular orbits. In this problem you will do the same for the more general case of elliptical orbits.



A planet of mass m orbits a star of mass M. M is much larger than m, so the star can be considered stationary. The orbit is an ellipse and the star is at one of its foci. The total energy of the system is E, which will be negative since the planet is "bound" to the star. Let the magnitude of the planet's angular momentum be ℓ.

1. In terms of E, ℓ, m, M, and G, find r_p and r_a (called, respectively, the perihelion and aphelion radii of the orbit). If you have difficulty, try Helping Questions 15, 16, and 17, one by one. **Key 21**

For every point on an ellipse, the sum of the distances from the point to the two foci is the same.

2. Use this fact to show that:

$$a = \frac{r_a + r_p}{2}$$
$$b = \sqrt{r_a r_p}$$

If you have trouble showing the second equation, use Helping Question 18.

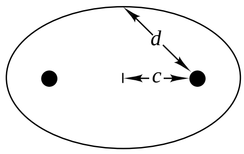
3. Express a and b in terms of E, ℓ, m, M, and G. **Key 3**

Kepler's second law says that the rate at which area is "swept out" by the planet's orbit is $\ell/2m$.

4. Use the second law to find the period of the orbit in terms of G, M, and a only. Stuck? Use Helping Questions 19 and 20. **Key 25**

Helping Questions

1. Can you combine Newton's second law and Newton's law of gravitation into one equation that relates the moon's acceleration a to M, G, and R? **Key 14**
2. What is the moon's acceleration a in terms of R and T? **Key 1**
3. Can you combine Newton's second law and Newton's law of gravitation to get an equation relating the given variables? **Key 22**
4. What would be the orbital period of such a satellite? **Key 16**
5. What is the orbital radius for a satellite circling the earth with period T? **Key 23**
6. If the masses of the stars were different, would the center of mass be fixed throughout the motion drawn in the diagram? **Key 17**
7. What is the force on the black hole? **Key 29**
8. What is the acceleration of the black hole? **Key 30**
9. What *two* physical quantities are conserved? **Key 10**
10. What is the total energy of the system when the meteor is far away? At its closest point? Use m for the mass of the meteorite and v for the speed of the meteorite at its point of closest approach. **Key 18**
11. What is the angular momentum of the meteor about the sun when it is far away? When it's at its closest point? **Key 5**
12. What is the kinetic energy of the system in terms of m, d, and ω? **Key 20**
13. What is the acceleration a of each mass in terms of ω and d? **Key 8**
14. What is the net force on each mass in terms of G, m, and d? **Key 31**
15. At any given time, let r be the distance to the planet from the sun, let v_r be the radial component of the velocity, and let $v_\perp$ be the component of the velocity which is perpendicular to the radius vector. In terms of these parameters and others, write an expression for E and an expression for ℓ. **Key 11**
16. At perihelion and aphelion, what is the value of v_r? **Key 33**
17. Taking for v_r the value just found, the two equations you wrote down in Helping Question 15 can be solved simultaneously for r. How many solutions are there? What do the solutions mean? **Key 7**
18. What are the lengths c and d in terms of r_a and r_p? Now use the Pythagorean theorem. **Key 19**



19. What is the orbital period in terms of the area A of the ellipse, ℓ, and m? **Key 6**
20. What is the area of the ellipse in terms of a and b? **Key 27**

Notes: Conserved Quantities Again

The notes for Learning Guide 3 said that if you can solve a problem by conservation principles, it's probably easier that way than by $\mathbf{F} = m\mathbf{a}$. On the other hand, for some problems the conservation laws by themselves simply don't give you enough information to solve the problem. Then you have to use $\mathbf{F} = m\mathbf{a}$. Let's look at three examples from this Learning Guide.

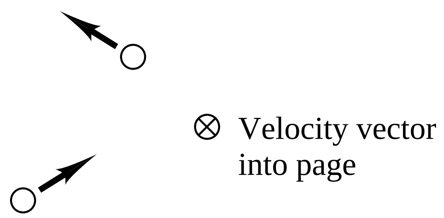
The only way to solve Problem IV is by using both conservation of energy and conservation of angular momentum. If you wanted, you could write down the differential equations that you get from the components of $\mathbf{F} = m\mathbf{a}$. They're just:

$$\frac{d^2x}{dt^2} = GMm\frac{x}{(x^2+y^2)^{3/2}}$$
$$\frac{d^2y}{dt^2} = GMm\frac{y}{(x^2+y^2)^{3/2}}.$$

But these equations would require advanced techniques to solve.

Notice that slight changes to the problem put the solution out of reach of conservation principles. Suppose the problem asked how much *time* it would take for the meteor to go from its initial point to its point of closest approach. Then you couldn't do the problem using only conservation laws; since time doesn't appear in the equations that the conservation laws give you, you'd have to use the *differential equations* written above. Or suppose the problem asked for the *angle* that the meteor's final velocity makes with its initial velocity. Once again, you'd have to use the differential equations written above.

Problem V is another instructive case. You solved this problem by using forces. However, the only reason you were able to avoid using a differential equation was that the motion was simple. If the particles had the initial velocities shown on the right, there is no way you'd be able to calculate how the kinetic energy and the potential energy vary with time. You could write down the differential equation governing the motion, but you wouldn't be able to solve it. (No one has ever solved this differential equation.)



When you get to thermodynamics you'll see more applications of conservation laws.

ANSWER KEY

1.
$$a = \left(\frac{2\pi}{T}\right)^2 R$$

2.
$$-\left(\frac{Gm_1m_2}{d} + \frac{Gm_2m_3}{d} + \frac{Gm_1m_3}{d}\right)$$
$$= -3\frac{Gm^2}{d}$$

3.
$$a = \frac{-GMm}{2E}; \quad b = \frac{\ell}{\sqrt{-2Em}}$$

4. $3Gm^2/2d$
5. $mv_0d; \quad mvb$
6. $T = 2mA/\ell$
7. Two — the smaller one is r_p and the larger one is r_a.
8.
$$a = \frac{\omega^2 d}{\sqrt{3}}$$

9.
$$b = \frac{-GM + \sqrt{(GM)^2 + v_0^4 d^2}}{v_0^2}$$

10. Total energy and angular momentum about the sun
11.
$$E = -\frac{GMm}{r} + \frac{1}{2}m(v_\perp^2 + v_r^2); \quad \ell = mv_\perp r$$

12. Yes:
$$\sqrt[3]{\frac{GMT^2}{4\pi^2}} - R_e \simeq 36000 \text{ km}$$
above a point on the equator, where $R_e \simeq 6400$ km is the radius of the earth.
13.
$$m_b = \frac{2\pi^2 d^3}{T^2 G}$$

14.
$$a = \frac{GM}{R^2}$$

15. $G = gr^2/M$
16. 24 h
17. No
18.
$$\frac{1}{2}mv_0^2; \quad \frac{1}{2}mv^2 - \frac{GMm}{b}$$

19.
$$c = \frac{r_a - r_p}{2}; \quad d = \frac{r_a + r_p}{2}$$

20.
$$K = \frac{1}{2}(3m)\omega^2\left(\frac{d}{\sqrt{3}}\right)^2 = \frac{1}{2}m\omega^2 d^2$$

21.
$$r_p = \frac{-GMm + \sqrt{(GMm)^2 + 2E\ell^2/m}}{2E}$$
$$r_a = \frac{-GMm - \sqrt{(GMm)^2 + 2E\ell^2/m}}{2E}$$

22. $g = GM/r^2$
23.
$$\sqrt[3]{\frac{GMT^2}{4\pi^2}}$$

24. $m_s = m_b$
25.
$$\frac{2\pi}{\sqrt{GM}} a^{3/2}$$

26.
$$\frac{4.0 \times 10^{14}}{M} \ \frac{\text{m}^3}{\text{s}^2}$$

27. $A = \pi ab$
28.
$$\frac{4.0 \times 10^{14}}{M} \ \frac{\text{m}^3}{\text{s}^2}$$

29.
$$\frac{Gm_s m_b}{d^2}$$

30.
$$a = \frac{d}{2}\left(\frac{2\pi}{T}\right)^2$$

31.
$$F = \frac{\sqrt{3}Gm^2}{d^2}$$

32.
$$G = \frac{4\pi^2 R^3}{T^2 M}$$

33. $v_r = 0$

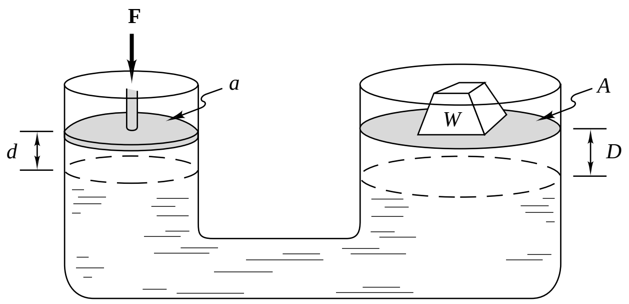
Princeton University
Physics 103
Fall Term 2004

Learning Guide No. 8: Fluid Mechanics

Suggested Reading: Tipler, Chapter 13

Problem I

In the *hydraulic press* drawn in the figure, the small piston on the left has cross-sectional area a and the larger piston on the right has area A. The fluid used in the press is water.



1. What force F must be applied to the small piston to hold a weight W on the large piston at the same level as the small piston? See Helping Question 1 if you need assistance. **Key 17**

2. If the force moves the left piston down a distance d, by what distance D does the weight rise? **Key 14**

3. At the levels reached after the motion of part (2), what force F' must be applied to the small piston to hold up the weight? Denote the mass density of water by ρ. Use Helping Question 2 if your answer doesn't check. **Key 18**

4. In a typical hydraulic press, $a = 1$ in^2 and $A = 1$ ft^2. The *weight* of water is about 60 lb/ft^3. Give numerical answers to parts (1), (2), and (3), assuming $W = 1440$ lb and $d = 5$ ft. **Key 21**

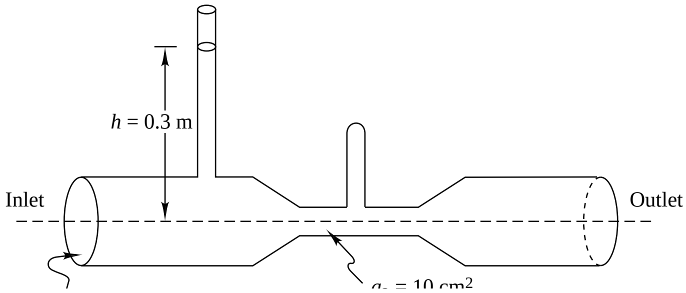
Problem II

A helium-filled balloon is anchored by a string to the floor of an enclosed railroad car. The train accelerates in the forward direction with acceleration $a = 1$ m/s^2.

1. Which way does the balloon tilt? Stumped? See Helping Question 3. **Key 6**

2. What angle does the string holding the balloon make with respect to the vertical? See Helping Question 4 for a hint. **Key 3**

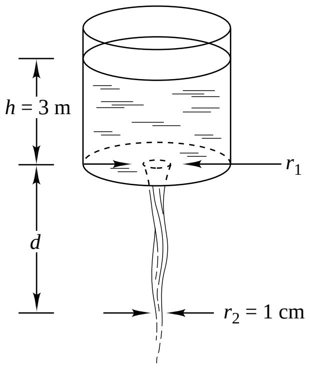
Problem III



The diagram on the right shows a Venturi device, which is being used to create a partial vacuum. The cross section decreases from $a_1 = 20$ cm^2 at the inlet and outlet to $a_2 = 10$ cm^2 at the narrow throat in the middle. The height of the water in the inlet standpipe, which is open to the atmosphere at its top, is $h_1 = 0.3$ m. The standpipe at the throat is closed and the pressure at the center of the throat is measured to be $p_2 = 0.9$ atm. You may neglect viscous forces and take atmospheric pressure to be $p_0 = 10^5$ Pa. How much water is flowing through the pipe (i.e., what is the volume flow rate, R, through the pipe)? If you need to, consult Helping Questions 5, 6, and 7. **Key 1**

Problem IV

Water flows through a hole at the bottom of a tank that is filled to a height $h = 3$ m, as shown. The radius of the hole is $r_1 = 1.5$ cm.

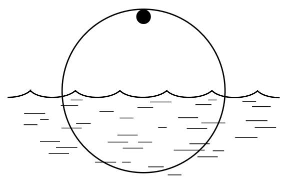


1. What is the speed of the water immediately after it leaves the hole? See Helping Question 8 if you need to. **Key 11**

2. At what distance d below the bottom of the tank is the radius of the stream reduced to $r_2 = 1$ cm? For a clue, see Helping Questions 9 and 10. **Key 25**

Problem V

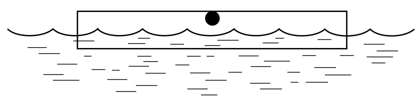
A plastic ball has a constant density equal to one-quarter that of water. Embedded in the surface of the ball is a solid steel pellet that has mass equal to that of the rest of the ball. The ball is put into a body of water, halfway immersed, with the pellet at its highest point, as shown.



1. Make a sketch that shows the center of mass and the center of buoyancy of the ball. Show also the gravitational force and the buoyant force on the ball and their effective points of application. Reread Section 13-3 of Tipler if you're confused. **Key 28**

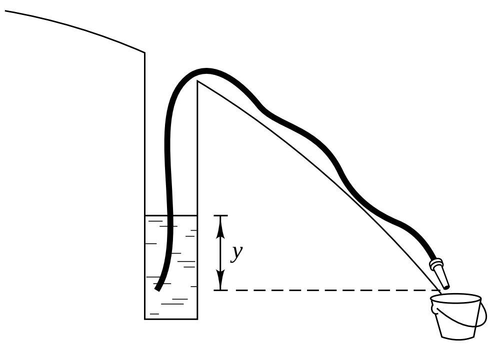
2. Your answer for part (1) shows that the ball is in static equilibrium. Is the equilibrium stable? **Key 7**

A solid barge is made of the same plastic and also has a small steel pellet embedded in its surface with mass equal to that of the entire rest of the barge. The barge is put into the water as shown.



3. Does the barge flop over the way the ball does? If you're just guessing, look at Helping Questions 11 and 12. **Key 2**

Problem VI



A boy is tired of hauling water pail by pail from a 20-ft well. Fortunately for him, the well is on the side of a hill, so he can hook up a siphon. He buys a long garden hose with an interior diameter of $d = 0.5$ in.

1. Having primed the siphon (filled it with water), the boy puts the free end of the hose $y = 5$ ft below the level of water in the well. How many seconds does it take him to fill his 5-gal pail? A gallon is about 200 in^3. Neglect the viscosity of the water. For a hint, see Helping Question 13. **Key 20**

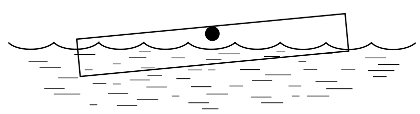
Note: By ignoring viscosity of the water, we have considerably underestimated the time required to fill the pail.

2. Having filled his pail, the boy foolishly throws the hose back up the hill so it's above the level of the water in the well. What happens? **Key 26**

3. Just for fun, can you think of a way to leave the hose in between uses, so that (a) water doesn't flow out of the well, (b) the siphon doesn't lose its prime, and (c) conditions (a) and (b) hold even if the surface level has large fluctuations? **Key 12**

HELPING QUESTIONS

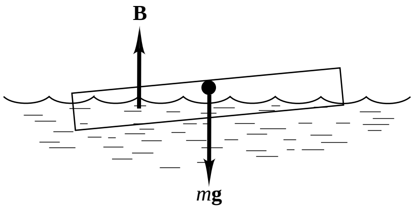
1. What physical quantity is a function only of depth in stationary liquids? **Key 4**
2. If the vertical separation between two points in a stationary liquid is h, what is the difference of the pressures at the two points? **Key 9**
3. Which way does the rest of the "heavy" air tend to move with respect to the accelerating car? **Key 10**
4. What does Einstein's equivalence principle say about the apparent direction of gravity inside the car? **Key 23**
5. What does Pascal's principle yield for the pressure at the bottom of the inlet standpipe? **Key 8**
6. What does the equation of continuity give for the ratio of the fluid speeds in the inlet (v_1) and in the throat (v_2)? **Key 15**
7. What other relation holds between the pressures and flow speeds at the inlet and throat portions of the device? **Key 22**
8. What does Bernoulli's equation tell you about the relationship between the speed of the stream and the height of the water? **Key 16**
9. How are the stream radius and the flow speed related? **Key 24**
10. What is the equation for the speed of a freely falling mass? **Key 19**
11. When the ball is rotated, always remaining half immersed, does its center of buoyancy change? What about when the barge rotates? **Key 13**
12. Suppose the barge had tipped over a little bit, as shown in the diagram. Indicate the gravitational force and the buoyant force and their effective points of application. Does the net external torque tend to right the barge or flop it over? **Key 5**



13. Consider points at the surface of the well water (outside the hose) and at the opening of the hose where water is coming out. What is the pressure at these two points? Now write Bernoulli's equation for these two points, assuming that the well diameter is much larger than the hose diameter. **Key 27**

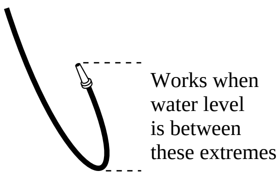
ANSWER KEY

1. $R = 5.87 \times 10^{-3}\ \mathrm{m^3/s} = 5.87\ \mathrm{L/s}$
2. No
3. $\theta = \tan^{-1}(a/g) = 5.8°$
4. Pressure
5.

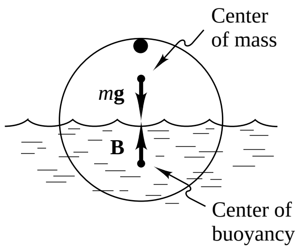


The torque tends to right the barge.

6. Forward
7. No
8. $p_1 = p_0 + \rho g h_1$
9. $\rho g h$
10. Backward
11. $v_1 = \sqrt{2gh} = 7.7\ \mathrm{m/s}$
12. Bend the hose so the last 6 or 8 ft runs back up the hill.



13. No; yes
14. $D = (a/A)d$
15. $v_1/v_2 = a_2/a_1$
16. $p_0 + \rho v_1^2/2 = p_0 + \rho g h$; $p_0 = 1$ atm (i.e., at both top *and* bottom)
17. $F = (a/A)W$
18.
$$F' = \frac{a}{A}W + \rho g d a \left(1 + \frac{a}{A}\right)$$
19. $v_2^2 = v_1^2 + 2g(y_1 - y_2) = v_1^2 + 2gd$
20. 23.7 s
21. $F = 10$ lb, $D = \frac{5}{12}$ in, $F' = 12.1$ lb
22.
$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$
23. There is an apparent acceleration due to gravity given by $g' = \sqrt{g^2 + a^2}$ that points down and slightly backward.
24. $v_1 \pi r_1^2 = v_2 \pi r_2^2$
25. $d = h\left[(r_1/r_2)^4 - 1\right] = 12.2$ m
26. The water in the hose goes back into the well.
27. $p_1 = p_2 = 1$ atm, ignoring the small change in atmospheric pressure over 5 ft. Thus, Bernoulli's equation
$$p_1 + \rho g y = p_2 + \frac{1}{2}\rho v_2^2$$
becomes
$$\rho g y = \frac{1}{2}\rho v_2^2,$$
where v_2 is the speed of the water as it comes out of the hose.
28.



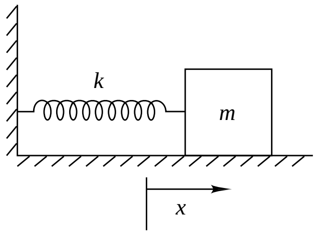
Princeton University
Physics 103
Fall Term 2004

Learning Guide No. 9: Oscillations

Suggested Reading: Tipler, Chapter 14

Problem I

An ideal spring of spring constant k is attached to a stationary wall on one end and a block of mass M on the other. The mass is pulled a distance A to the right of its equilibrium position and released from rest. Assume there is no friction between the table and the block.



1. What is the angular frequency ω of the system in terms of k and m? If you don't know, take another look at Tipler. Otherwise, use the text as little as possible for the rest of this problem. **Key 26**

2. What is the position of the block $x(t)$ as a function of time? Express your answer in terms of A, ω, and t. Take the zero of the x-axis to be at the equilibrium point of the spring and the positive direction toward the right. **Key 5**

3. What is the block's velocity $v(t)$? Express your answer in terms of A, ω, and t. **Key 31**

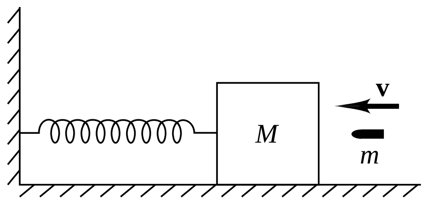
4. What is the system's potential energy $U(t)$? Write your answer in terms of A, k, ω, and t. **Key 10**

5. What is the block's kinetic energy $K(t)$? Express your answer in terms of A, m, ω, and t. **Key 21**

6. Verify that the total energy $E(t)$ of this system is constant. **Key 16**

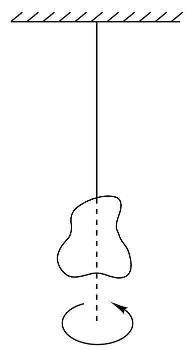
Problem II

The system of Problem I is at rest at equilibrium. A bullet of mass m moving with velocity v embeds itself in the block. Assume that the collision between the bullet and the block takes place in such a short time interval that during the collision the spring compresses by a negligible amount. The block will, however, be set in motion by the bullet's impact. What is the amplitude A of the resulting motion? Use Helping Questions 1 and 2. **Key 33**



Problem III

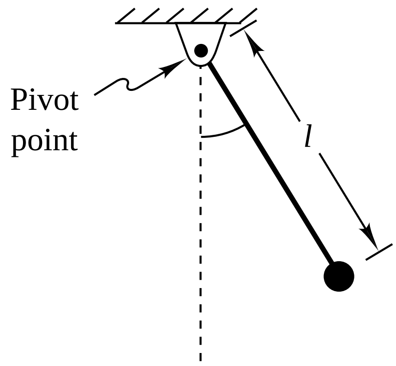
An object is hung by a wire from the ceiling. The object has rotational inertia I about the axis made by the extended wire. When the object is twisted through an angle θ, the wire exerts a torque τ that tends to restore the system to equilibrium. The scalar dependence of τ on θ is Hooke's law $\tau = -\kappa\theta$, where the minus sign indicates the restoring nature of the torque and κ is a constant for the wire (analogous to k for a linear spring).



1. Write down a differential equation that the motion $\theta(t)$ must obey. Helping Question 3 should be enough if you're confused. **Key 1**
2. Write down the general solution to this equation. For assistance, see Helping Question 4. **Key 6**
3. If the wire is shortened by a factor of 2, will the period be larger or smaller? See Helping Questions 5 and 6. **Key 22**

Problem IV

The pendulum drawn in the diagram swings without friction under the influence of gravity. Assume that the rod holding the body is massless and rigid and doesn't stretch.



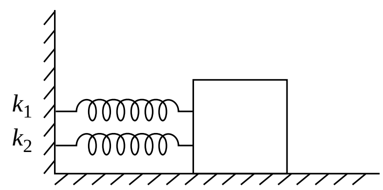
1. Write down a differential equation that the motion $\theta(t)$ must satisfy exactly. You may refer to Helping Question 7. **Key 42**

2. Approximate this differential equation for the case of small amplitudes by a differential equation you can solve. Solve it. Use Helping Question 8 if you can't think of a good approximation. **Key 23**

3. If the pendulum's length is made shorter, does its period become larger or smaller? **Key 9**

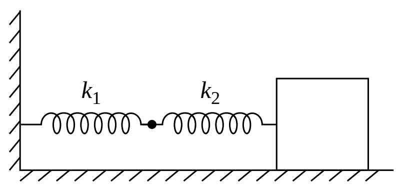
Problem V

In the situation shown in the sketch, the springs are ideal and the surfaces frictionless. Both springs have the same equilibrium extension. The mass moves in simple harmonic motion with angular frequency ω. Suppose that the two springs are replaced by a single spring with spring constant k and the mass moves with the same angular frequency ω.



1. What must k be in terms of k_1 and k_2? If you're lost, see Helping Question 9. **Key 18**

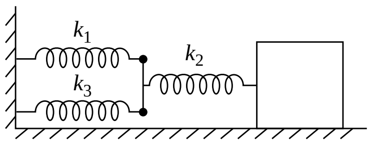
In a second situation, the same two springs are put into the second configuration shown. The mass moves with angular frequency ω'. Again suppose that the two springs are replaced by a single spring with spring constant k' and the mass moves with the same angular frequency ω'.

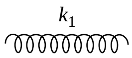


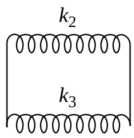
2. What must k' be in terms of k_1 and k_2? Refer to Helping Questions 10 and 11 if you need help. **Key 39**

Thus with two springs, you can make four "effective springs," with spring constants k_1 (the first spring alone), k_2 (the second spring alone), k (the springs "in parallel"), and k' (the springs "in series").

3. Just for fun, how many effective springs can you make with three springs? You are allowed to have series, parallel, and series-parallel (shown on the right) combinations. Assume that k_1, k_2, and k_3 are chosen so that there are no "accidental equivalences" such as:

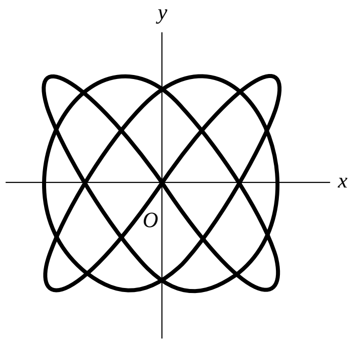


$k_1 = k_2 + k_3$ making 



(**Hint:** Draw pictures!) **Key 44**

Problem VI



The curve in the sketch is the path of a particle executing simple harmonic motion both horizontally and vertically: its path is given by the equations:

$$x(t) = A_x \cos(\omega_x t + \phi_x)$$

$$y(t) = A_y \cos(\omega_y t + \phi_y)$$

1. For the motion shown in the diagram, what is the ratio of the periods T_x/T_y? See Helping Question 12 if you don't know how to get started. **Key 29**

The motion shown in the picture is periodic: the particle returns to its initial position with its initial velocity after a certain time T and then simply repeats its motion forever.

2. For what values T_x/T_y is the motion periodic? Need help? See Helping Question 13. **Key 13**

Problem VII

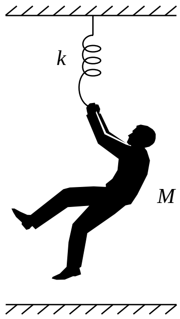
A block of mass m hangs from a spring of spring constant k. The block is pulled down a distance A from equilibrium and released from rest. The damping constant of the oscillator is b and so the block moves according to:

$$x = Ae^{-bt/2m}\cos(\omega' t),$$

where ω' is a number slightly less than $\sqrt{k/m}$.

1. When will the amplitude be $A/2$? Use Helping Question 14. **Key 12**
2. Roughly what fraction of the original energy will have been lost by this time? **Key 3**

Problem VIII



A malicious man wants to destroy a strong spring by stretching it beyond its elastic limit. The spring is one meter long at equilibrium and has an elastic limit of 0.5 m. If it is stretched past 1.5 m or compressed past 0.5 m, it will not regain its original shape and will be ruined. In between these extremes, it is an ideal spring of spring constant 10,000 N/m. The man has a mass of 100 kg.

1. Does the man succeed in destroying the spring by hanging from it? **Key 4**

The man drops back to the ground to look for paper and pencil. Next he plans to destroy the spring by delivering a force to the system that consists of the spring and his body. He plans to vary the force sinusoidally in time at precisely the resonant frequency of the spring-body system. He writes down the following equation (which is correct and graphed in Figure 14-24 of Tipler):

$$A = \frac{F_m}{\sqrt{m^2(\omega_e^2 - \omega^2)^2 + b^2\omega_e^2}}.$$

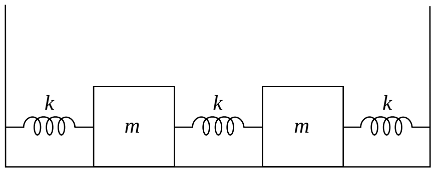
In this equation, F_m is the maximum value of the sinusoidally varying driving force, ω_e is the angular frequency of the driving force, ω is the natural angular frequency of the spring-body system, and b is the damping constant of the system. A is the amplitude of the resulting motion.

2. He estimates the damping constant at 1 kg/s. He plans to set ω_e equal to ω. How many cycles per second is this, and how many newtons must F_m be so that $A =$ 0.5 m? **Key 15**

The man jumps back on the spring to put his idea into practice. He simulates the external force by slightly oscillating his body. But he can't quite oscillate with a frequency $\nu = 1.6$ Hz. He does manage to oscillate with $\nu = 1$ Hz.

3. Calculate roughly what F_m has to be for A to be 0.5 m at this lower driving frequency. Does he succeed in destroying the spring? **Key 20**

Problem IX



In this problem you will examine an example of *coupled oscillations.* Consider two blocks of equal mass and three identical springs arranged as shown in the figure. Ignore friction throughout this problem.

1. If x_1 and x_2 represent the displacement from the equilibrium position of the respective blocks, show that:

$$m\frac{d^2x_1}{dt^2} = k(x_2 - 2x_1) \qquad \text{and} \qquad m\frac{d^2x_2}{dt^2} = k(x_1 - 2x_2)$$

Key 14

2. Find all the values of ω for which there are values of A_1 and A_2 such that:

$$x_1(t) = A_1 \sin(\omega t)$$
$$x_2(t) = A_2 \sin(\omega t)$$

is a solution to the above pair of differential equations. If you can't find a good method of attack, see Helping Question 15. **Key 30**

What is the physical meaning of the two angular frequencies you found in part (2)? In parts (3) and (4) you will discover the answer to this question.

3. Imagine that the middle spring is replaced by a massless, inextensible rod so that the two masses oscillate back and forth together ($A_1 = A_2$). What is their angular frequency of oscillation? Use Helping Question 16 if your answer doesn't check. **Key 8**

4. Now imagine that the middle spring is back in place but held fixed at its center. The masses oscillate symmetrically toward and away from each other ($A_1 = -A_2$). What is their angular frequency of oscillation? Use Helping Question 17 if you need help. **Key 43**

If you take a linear algebra course you will learn a more systematic approach to part (2) that generalizes to more springs and more blocks with different spring constants and masses. In the language of linear algebra, the heart of the method is finding the *eigenvalues* and *eigenvectors* of a *symmetric matrix.*

HELPING QUESTIONS

1. Is momentum conserved during the collision? Then what is the velocity $v_{\max}$ of the block immediately after the collision? **Key 17**
2. Is energy conserved after the collision? Can you get an equation relating the amplitude of the resulting motion to the block's velocity just after the collision? **Key 11**
3. What is Newton's law for rotational motion? Can you combine it with Hooke's law to get a differential equation for θ? **Key 35**
4. You know that the general solution of
$$m\frac{d^2x}{dt^2} = -kx \quad \text{is} \quad x(t) = A\cos\left(\sqrt{\frac{k}{m}}t + \phi\right).$$
Does this help? **Key 28**
5. What is the torsional pendulum's period T in terms of κ and I? **Key 38**
6. Use your physical intuition to decide whether κ would increase or decrease when the wire is shortened. **Key 25**
7. Draw a force (free-body) diagram for the mass. What is the net tangential force? Now use $\tau = I\alpha$. **Key 27**
8. For small θ, what is $\sin\theta$ approximately equal to? **Key 37**
9. In each case, if the mass is displaced a distance x, what is the force on the mass? **Key 41**
10. If the force on the mass is F, what are the magnitudes of the two forces acting on spring 2? The two forces acting on spring 1? **Key 32**
11. If the force on the mass is F, what is the extension x_1 of spring 1? The extension x_2 of spring 2? Can you express $F/(x_1 + x_2)$ in terms of k_1 and k_2? **Key 24**
12. Pick an arbitrary point along the path to serve as your starting point. How many cycles n_x in the x-direction does the particle make before returning? How many cycles n_y in the y-direction? Suppose the total time around the path is T. Express T in terms of n_x and T_x, and also in terms of n_y and T_y. **Key 2**
13. What conditions on ω_x and ω_y make the motion periodic with period T? Find an expression for T_x/T_y in terms of n_x and n_y. **Key 19**
14. Which factor of the equation $x = Ae^{-bt/2m}\cos(\omega' t)$ has to do with the decrease in amplitude? What is the value of this factor when the amplitude is $A/2$? **Key 7**
15. Plug in the conjectured solution, differentiate, and eliminate the time t to yield two algebraic equations. Eliminate ω temporarily to get a quadratic equation relating A_1 to A_2. What is this quadratic equation? Use the relations between A_1 and A_2 that this quadratic equation gives you to find the possible values of ω. **Key 34**
16. Use what you learned in Problem V about combining springs. What is the effective spring constant? What is the total mass? **Key 36**

17. If a spring is cut in half, what happens to its spring constant? Now use what you know about combining springs. **Key 40**

Notes: Differential Equations

You have probably noticed that this Learning Guide has a slightly different character from the first seven Learning Guides. There is a reason for this — the subject matter has changed slightly. *You have now learned the principles of classical mechanics, and are moving on to its applications!*

Earlier on you studied the basic principles of mechanics — Newton's second law ($\mathbf{F} = m\mathbf{a}$ for a single particle) and Newton's third law. As time went on you studied some direct implications of these two laws — conservation of energy, Newton's second law for systems of particles ($\mathbf{F}_{\text{ext}} = M\mathbf{a}_{\text{CM}}$), conservation of momentum, and Newton's second law for rotational motion ($\boldsymbol{\tau} = d\mathbf{L}/dt$). Starting with Learning Guide 8, you study applications of these principles — simple harmonic motion, fluid mechanics and waves. As you've learned in this Learning Guide, applying Newton's laws is easier said than done. The heart of the difficulty is that differential equations occur naturally, and some are hard to solve. How do differential equations get into physics? Because Newton's second law itself is a differential equation:

$$F = ma = m\frac{d^2x}{dt^2}.$$

Let's compare solutions to this differential equation for three different force laws:

Constant force F	Hooke's law $F = -kx$	Pendulum $F = mg\sin\theta$
$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{F}{m}$	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{k}{m}x$	$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -(g/\ell)\sin\theta$
$\downarrow$	$\downarrow$	$\downarrow$
$\frac{\mathrm{d}x}{\mathrm{d}t} = v_0 + \frac{F}{m}t$		
$\downarrow$	$\downarrow$	$\downarrow$
$x = x_0 + v_0t + \frac{1}{2}\frac{F}{m}t^2$	$x = A\sin\left(\sqrt{\frac{k}{m}}t + \phi\right)$	? ? ? ?

When you first learned $\mathbf{F} = m\mathbf{a}$, the problems you were asked to do involved constant forces. Although you may not have realized it at the time, you already knew the solution to the resulting equation of motion — i.e., you knew that for constant acceleration in the x direction the resulting displacement is

$$x(t) = x_0 + v_{0x}t + \frac{1}{2}at^2.$$

Hooke's law, however, leads to a more difficult differential equation to solve; in effect, Tipler "solved" it by just guessing the right solution! To see that their guess is indeed correct, you need only plug it back into the original differential equation. Had their guess been wrong, it would have led to a contradiction. The guess is a perfectly acceptable and time-honored method of solving differential equations, but it has an obvious limitation. If you take a course in differential equations, you will learn a more systematic way to solve this differential equation, as well as the differential equations of damped and forced oscillations:

$$m\frac{d^2x}{dt^2} + kx + b\frac{dx}{dt} = 0 \qquad \text{and} \qquad m\frac{d^2x}{dt^2} + kx + b\frac{dx}{dt} = F_m \cos(\omega_e t).$$

Most differential equations in physics are more like the pendulum equation — they are either very difficult or actually impossible to solve exactly. So techniques for finding approximate solutions are extremely important.

Here are some questions to think about, relating to approximations in three of the problems you've just finished. The answers are at the end of the notes.

Problem IV: How well does the approximate solution to the pendulum equation

$$\theta(t) = A\sin\left(\sqrt{\frac{g}{\ell}}t + \phi\right)$$

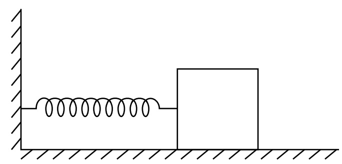
describe the real solution? Are the two periods similar? For the same initial conditions, would $\theta_{\text{approx}}(t) \simeq \theta_{\text{real}}(t)$ for large values of t?

Problem VI: If the motion had been given by the differential equations

$$\frac{d^2x}{dt^2} = -k_1 \sin x \quad \text{and} \quad \frac{d^2y}{dt^2} = -k_2 \sin y,$$

could you have used methods of approximation to decide when the motion was periodic?

Problem VIII: Do you think the force of friction is well approximated by $-b\mathrm{d}x/\mathrm{d}t$ in the diagram? When you study electricity and magnetism you will become an expert in turning impossible problems into solvable problems by making small approximations.



One thing that decreases the difficulty caused by differential equations is that, often in physics, the same differential equation occurs in very different physical contexts. In particular, the differential equation for simple harmonic motion applies very often, especially if one is only interested in the approximate behavior of a system. As you saw in Problems III and IV, differential equations are no problem if you already know how to solve them! As you study more physics, you will spend more time learning how to solve and intuitively understand the differential equation for forced oscillations in a variety of contexts. For example, you will see that the equations for harmonic oscillators also apply to electrical oscillations.

Answers to the Preceding Questions:

IV: For small oscillations, the periods are similar but $\theta_{\text{approx}}(t)$ does not approximate $\theta_{\text{real}}(t)$ for large values of t (the two will be on different sides of equilibrium half the time).

VI: No. Even if the x and y motions are periodic, the overall motion is periodic only if the ratio of the periods is rational. This can't be decided with approximate methods.

VIII: It would be a poor approximation. As you know, a better approximation for the force of friction is $\mu_k N$ — the frictional force depends only on the direction of the velocity, not its magnitude.

ANSWER KEY

1.
$$I\frac{d^2\theta}{dt^2} = -\kappa\theta$$

2. Three; four; $T = n_x T_x = n_y T_y$

3. About $\frac{3}{4}$. The energy is not exactly $k/2$ times the amplitude squared; it is sometimes slightly above or below this value. You can see this by looking at the power $\mathbf{F}\cdot\mathbf{v}$ lost to friction.

4. No. He just stretches it 10 cm.

5. $x(t) = A\cos(\omega t)$

6.
$$\theta(t) = A\cos\left(\sqrt{\frac{\kappa}{I}}t + \phi\right)$$

7. $e^{-bt/2m}$; it equals $\frac{1}{2}$.

8. $\omega = \sqrt{k/m}$

9. Period becomes smaller.

10.
$$U(t) = \frac{1}{2}kA^2\cos^2(\omega t)$$

11. Yes.
$$\frac{1}{2}(M+m)v_{\max}^2 = \frac{1}{2}kA^2$$

12.
$$t = \frac{2m\ln 2}{b}$$

13. The motion is periodic when T_x/T_y is a rational number. It is not periodic when T_x/T_y is an irrational number.

14. Draw a force (free-body) diagram of each spring and block. Then use Newton's law and Hooke's law.

15. About 1.6 Hz; 5 N

16.
$$\begin{aligned} E(t) &= U(t) + K(t) \\ &= \frac{1}{2}kA^2\cos^2(\omega t) + \frac{1}{2}mA^2\omega^2\sin^2(\omega t) \\ &= \frac{1}{2}kA^2\cos^2(\omega t) + \frac{1}{2}kA^2\sin^2(\omega t) \\ &= \frac{1}{2}kA^2 \end{aligned}$$

17. Yes: since the spring compresses only by a negligible amount during the collision, there are essentially no external forces on the block-bullet system. Thus:
$$v_{\max} = \frac{m}{M+m}v$$

18. $k = k_1 + k_2$

19. $\omega_x T = 2\pi n_x$ and $\omega_y T = 2\pi n_y$, where n_x and n_y are *integers*. $T_x/T_y = n_y/n_x$

20. F_m would have to be about 3000 N, so he doesn't succeed.

21.
$$K(t) = \frac{1}{2}mA^2\omega^2\sin^2(\omega t)$$

22. Smaller

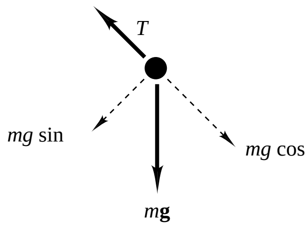
23.
$$\ell\frac{d^2\theta}{dt^2} \simeq -g\theta$$
$$\theta = A\cos\left(\sqrt{\frac{g}{\ell}}t + \phi\right)$$

24. F/k_1; F/k_2; yes: after you plug in for x_1 and x_2, the F's cancel.

25. Increase. For a given θ, the wire is twisted more *per unit length.* Thus, the torque that it exerts on the object is increased.

26. $\omega = \sqrt{k/m}$

27.



Net tangential force $= mg \sin\theta$.

28. It helps: the differential equation for the motion of a mass on a spring is the same as the differential equation for the motion of the torsional pendulum, except that the names of the variables are changed.

29. 4/3

30. $\sqrt{k/m}$; $\sqrt{3k/m}$

31. $v(t) = -A\omega \sin(\omega t)$

32. All four forces have magnitude F.

33.
$$A = v\sqrt{\frac{m^2}{(M+m)k}}$$

34. $A_1^2 = A_2^2$, so $A_1 = \pm A_2$.

35.
$$\tau = I\alpha = I\frac{d^2\theta}{dt^2}.$$

Yes: you now have two expressions for torque, so you can set them equal.

36. $2k$; $2m$

37. θ

38.
$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$

39.
$$k' = \frac{k_1 k_2}{k_1 + k_2}$$

40. It doubles.

41. $-k_1 x - k_2 x$; $-kx$

42.
$$\ell\frac{d^2\theta}{dt^2} = -g\sin\theta$$

43. $\omega = \sqrt{3k/m}$

44. 14 combinations + 3 original springs

Princeton University
Physics 103
Fall Term 2004

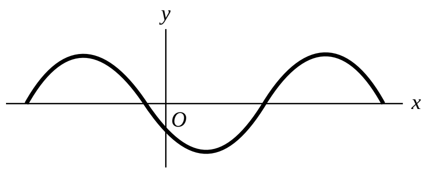
Learning Guide No.: 10 Waves and Sound

Suggested Reading: Tipler, Chapters 15 and 16 (Optional: Section 16-5)

Problem I

A wave travels on a hypothetical stretched string of infinite length. At time t, the element of the string at position x is displaced a distance:

$$y = A \ \sin(x - t),$$



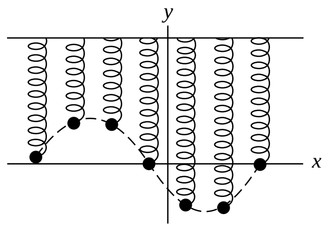
where $A = 1$ cm $= 0.01$ m, x has units of meters, and t has units of seconds. The graph shows the shape of the string at a fixed point in time (the scales of the two axes are different).

1. What is the angular frequency ω? The frequency f? If you have trouble with this part or parts (2) through (6) reread, Tipler, Section 15-2. **Key 9**
2. What is the wave number k? **Key 32**
3. What is the period T? **Key 10**
4. What is the wavelength λ? **Key 28**
5. What is the phase velocity v? **Key 37**
6. What is the amplitude y_m? **Key 30**
7. Is energy transferred to the left or to the right? If you can't answer this or the next part, reread Tipler, pp. 475-476. **Key 2**
8. What else do you need to know before you can calculate the amount of energy transferred per unit time? Does it make sense physically that two waves of the same size, shape, and speed can transfer different amounts of energy? **Key 16**

Problem II

Let the mass per unit length of the string in Problem I be μ.

The apparatus in the sketch is a collection of an infinite number of simple harmonic oscillators. The mass hanging from each spring is a segment of length $\Delta\ell$ of the string of Problem I. The spring constants and the initial positions and velocities are chosen so that the motion of the string is precisely what it was in Problem I.



1. What is the total energy contained in the oscillators per unit length of the string? Express your answer in terms of y_m, the frequency f, and μ. Turn to Helping Questions 1 and 2 if you're stuck. **Key 8**

2. One could define the "average rate of energy transfer" for the system of oscillators to be $P = Ev$, where E is the energy per unit of length calculated in part (1) and v is the phase velocity of the wave. But is energy really being transferred horizontally in the collection of oscillators? Was it really being transferred in the string of Problem I? **Key 27**

Problem III

The wave function of a guitar string vibrating without any overtones is

$$y(x,t) = 0.003 \times \sin(4x) \times \cos(2080\,t) \text{ m},$$

where x and y are in meters and t is in seconds.

1. Verify that this is a reasonable wave function by calculating the length of the string and the pitch of the tone produced. Use Helping Question 3 for the length of the string and Helping Question 4 for the pitch of the tone. **Key 20**

2. What are the phase velocities of the traveling waves that sum to the standing wave? See Helping Question 5 for a hint. **Key 34**

3. What is the maximum speed of the midpoint of the stretched string? Clueless? See Helping Question 6. **Key 23**

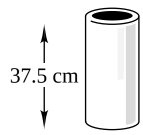
Problem IV

Since the wave motion of a guitar string is damped, the wave function given in Problem III is unrealistic. Indeed, the very fact that mechanical energy from the string is transformed into sound energy tells you that the string's motion must be damped.

1. The guitar string of Problem III vibrates with an *initial* amplitude of 3 mm. The mass per unit length of the string is 1 g/m. Assume first that there is no loss of energy to heat. About how many dB is the sound intensity one meter in front of the guitar? You will have to make some approximations and some educated guesses, but you should be within 15 dB of the Answer Key. Use Helping Questions 7, 8, 9, and 10. **Key 5**
2. Look at Table 15-1 in Tipler. Is your answer to part (1) reasonable? About what fraction of the mechanical energy is lost to heat? **Key 39**

Problem V

A 37.5-cm-long metal pipe is closed at one end. A small loudspeaker is placed near the open end. The speed of sound in the pipe is 333 m/s.



1. At what frequencies will resonance occur in the pipe as the sound frequency emitted by the speaker is varied from 200 to 1200 Hz? If your answer doesn't check with the one in the key, take another look at Section 16-2 of Tipler. **Key 3**
2. At what frequencies would resonance occur if both ends were open? **Key 25**

Problem VI

The ratio between the frequencies of any two adjacent keys on the piano is the same:

$$\frac{f_{\text{higher}}}{f_{\text{lower}}} = \sqrt[12]{2} = 1.06\,.$$

The lowest string on a piano has a frequency of 27.5 Hz, whereas the highest string on the piano has a frequency of 4186 Hz.

1. Using the criterion that beats can be detected by the ear up to frequencies of about seven per second, for which pairs of keys can you hear beats? See Helping Question 11. **Key 11**
2. Can you actually hear beats between the notes? **Key 17**

Problem VII

A woman is riding a train moving at 135 mi/h, which is one-fifth the speed of sound. As she approaches her friend who is standing by the tracks, she blows her trumpet with a frequency f.

1. What frequency does her friend hear? If you don't remember the Doppler equation, look again at Section 15-5 of Tipler. **Key 18**

On the return trip, the woman's friend, still standing by the tracks, blows his trumpet with frequency f.

2. What frequency does the woman in the train hear? **Key 1**
3. Why are your answers to parts (1) and (2) different — doesn't each person see the other approach at 135 mi/h? **Key 35**

Problem VIII

The general Doppler equation derived in Section 15-5 of Tipler has to be modified to describe the Doppler effect for *light* waves exactly. However, even unmodified it is a good approximation.

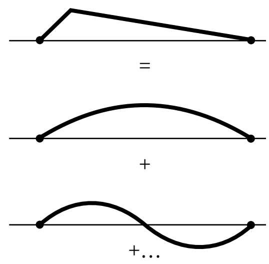
A physicist runs a red light. He contests the ticket and explains to the judge that, because of the Doppler effect, the light looked green to him. The judge declares him innocent of running the red light but fines him instead one dollar for every mile per hour he was moving above 60 mi/h.

How many dollars is the physicist fined? The speed of light is about $c = 186,000$ mi/s, and the wavelengths of red and green light are about 6.3×10^{-7} m and 5.3×10^{-7} m, respectively. If you can't remember the formula, refer to Section 15-5 of Tipler. **Key 36**

Problem IX

(Optional. You might want to try this if you're comfortable with integrals. You don't have to be an expert by any means.)

A stretched wire has fundamental frequency $f = \omega/2\pi$. When it is plucked, a listener hears not only the fundamental frequency, but also higher frequencies. In this problem you will learn the mathematics necessary to answer the question, "Exactly which harmonics are present and how strong are they?"



The secret of the answer lies in the superposition principle. Suppose the wire is pulled out near one of its ends as shown. As Tipler discusses in Sections 16-3, the function $f(x)$ can be decomposed in exactly one way into a sum of sine waves. That is, there is exactly one choice for the numbers $A_1, A_2, A_3, \ldots$ such that

$$f(x) = A_1 \sin x + A_2 \sin(2x) + A_3 \sin(3x) + \cdots$$

is true. Notice that once you know the *Fourier coefficients* $A_1, A_2, \ldots$ of the function $f(x)$, you know the strength of all the harmonics: the vertical displacement y of the wire at position x and time t is

$$y(x,t) = A_1 \sin x \cos(\omega t) + A_2 \sin(2x) \cos(2\omega t) + \cdots$$

So the nth harmonic has amplitude A_n.

How do you get the coefficients $A_1, A_2, \ldots$? Before answering this question, let's take a detour that may seem completely irrelevant right now. Define the *dot product of two functions* $f(x)$ and $g(x)$ to be the number

$$f(x) \cdot g(x) \equiv \frac{2}{\pi} \int_0^\pi f(x) g(x)\, dx$$

If you're a calculus whiz, you can do the integrals to check that the *dot product* of $\sin(n_1 x)$ and $\sin(n_2 x)$ equals 1 if $n_1 = n_2$ and 0 if $n_1 \neq n_2$. In other words, the functions $\sin x, \sin(2x), \ldots$ behave under their dot product the same as the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ behave under theirs.

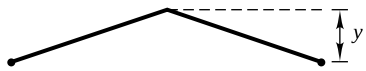
1. Without doing the integral, show that $\sin x \cdot \sin(2x) = 0$. Use Helping Question 12.

The point of introducing the dot product for functions is that the nth Fourier coefficient A_n of a function $f(x)$ is just $f(x) \cdot \sin(nx)$. You can see this very easily, since

$$\begin{aligned} f(x) &= A_1 \sin x + A_2 \sin(2x) + \cdots, \\ f(x) \cdot \sin(nx) &= [A_1 \sin x + A_2 \sin(2x) + \cdots] \cdot \sin(nx) \\ &= A_1(0) + A_2(0) + \cdots + A_n(1) + \cdots \\ &= A_n. \end{aligned}$$

This is completely analogous to the fact that you can get the components v_x, v_y, and v_z of a vector $\mathbf{v}$ by "dotting" $\mathbf{v}$ with $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

The wire is plucked by pulling its center a distance y from equilibrium and then releasing the wire from rest.



2. Without doing any integrals, what can you say about the amplitude of the harmonics $2\omega, 4\omega, 6\omega, \ldots$? After a good effort, look at Helping Question 13. **Key 14**

If you know how to "integrate by parts," you can get the complete solution.

3. For every n, find the exact value of A_n. If you need hints, see Helping Questions 14 and 15. **Key 31**

HELPING QUESTIONS

1. How many oscillators are there per unit length? What is the maximum extension of each oscillator? **Key 29**
2. What is the spring constant of each oscillator? **Key 13**
3. What must ℓ be if $\sin(4x) = 0$ at $x = 0$ and at $x = \ell$ but never equals zero in between? **Key 24**
4. If $\omega = 2080\,\text{rad/s}$, what is the frequency f? Is this a typical musical note? **Key 22**
5. What are the wave functions of the two traveling waves? What is the phase velocity of a traveling wave of angular frequency ω and wave number k? **Key 12**
6. What is $\sin(4x)$ at the middle of the string? Then what is dy/dt? **Key 26**
7. From Problem III, the midpoint of the string has maximum speed about 6 m/s. So a "typical" point on the string would have a maximum speed of about 3 m/s. Roughly, what is the initial energy of the string? **Key 19**
8. Estimate how long it takes for the string to lose half its energy. If the string loses energy at a constant rate, how many watts of sound are given off by the string? **Key 6**
9. Assume that no energy is given off behind the guitar and that energy is given off equally in all directions in front of the guitar. About how many watts are transferred through a square meter one meter away? **Key 33**
10. How many decibels is your answer for Helping Question 9? **Key 38**
11. If the higher-frequency string has frequency f_1 and the lower f_2, what is the frequency of their beats? **Key 40**
12. Graph the functions $\sin x$ and $\sin(2x)$ on the interval $0 < x < \pi$. Use these two graphs to graph their product function $\sin x \sin(2x)$ (not their dot product!) on the same interval. What is the net area under the curve? Use symmetry! **Key 4 & Key 7**
13. The key idea is contained in Helping Question 12.
14. You know that if n is even, $A_n = 0$. If n is odd, what relation holds between

$$\frac{2}{\pi}\int_0^{\pi/2} f(x)\,\sin(nx)\,dx \qquad \text{and} \qquad \frac{2}{\pi}\int_0^{\pi} f(x)\,\sin(nx)\,dx?$$

Key 21

15. You must integrate

$$2\left[\frac{2}{\pi}\int_0^{\pi/2} f(x)\,\sin(nx)\,dx\right],$$

which is

$$\frac{8y}{\pi^2}\int_0^{\pi/2} x\sin(nx)\,dx$$

In the usual notation of integration by parts, what should you take for u and what should you take for dv? **Key 15**

<u>Notes</u>: Orders of Magnitude

The problems in the Learning Guides that you've done so far have consistently de-emphasized arithmetic. As you recognize by now, knowing that the answer to a certain problem is, say, 6.5 and not 6.2, almost never adds to your understanding of the problem. However, knowing that the answer to a problem is 6×10^5 and not 6×10^2 does add to your understanding of the problem!

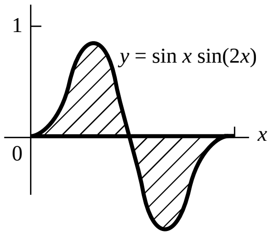
A rough indication of the size of a number is the power of 10 that most closely represents it. This exponent is called the number's order of magnitude. *You should know roughly the orders of magnitude of the quantities you're working with.*

This Learning Guide's notes are not suggesting that you should forget what you've learned and plug in numbers everywhere! Rather, they are advising you to try to *understand physically the magnitudes involved.* You obviously couldn't just say that the speeding physicist would be fined 120 million dollars, but you should recognize immediately that his defense is utterly ridiculous. When you check a numerical answer with the Answer Key, you should do more than just see that it matches — you should think about it for a little while and *make it part of your physical intuition.* If you were surprised that the phase velocity in a guitar string was more than half a kilometer per second, you shouldn't be surprised again! Of course, for an answer like "half a kilometer per second" to mean anything to you, you have to have some reference points. For example, you should have *no doubts* about the speed of sound being about a mile every five seconds, atmospheric pressure being about 14 lb/in^2, and water weighing about 60 lb/ft^3. To help build up your intuition about sound, you should look carefully at the problems in Chapter 15 of Tipler. You might be surprised!

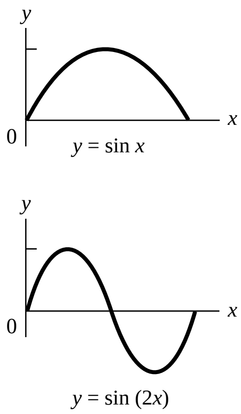
Physicists often call a rough calculation an "order-of-magnitude" or "back-of-the-envelope" calculation. Even though they're only approximate, they can be very revealing. Problem IV about the guitar string is a good example — the crudest of calculations show that almost all of the string's energy goes into heat.

ANSWER KEY

1. $1.20f$
2. To the right
3. 222, 666, and 1110 Hz
4. The two shaded regions have equal areas.



5. About 10^{-4} W/m^2, or 80 dB; the same as a jackhammer 10 feet away
6. About 5 s; about 5×10^{-4} W
7.



8. $2\pi^2 y_m^2 f^2 \mu$
9. $\omega = 1$ rad/s; $f = 1/2\pi$ Hz
10. 2π s
11. Every adjacent pair among the bottom 27 keys (plus some keys that are close but not adjacent in the very lowest octave)
12. $\sin(\omega t - kx)$ and $\sin(\omega t + kx)$; ω/k
13. $\omega^2 \mu \, \Delta\ell$
14. They're zero.
15. $u = x$; $dv = \sin(nx)\,dx$
16. The mass per unit length of the string (or the tension in the string); a wave on a thick rope (μ large) will transfer more energy than a wave with the same wave function on a thin string (μ small).
17. The intensity of the sound goes up and down, but there are no clearly defined beats since the tones aren't pure.
18. $1.25f$
19. About 0.005 J
20. 78.5 cm; 331 Hz
21. The second is twice the first.
22. 331 Hz; yes
23. 6.24 m/s (vertically)
24. $\ell = \pi/4$ meters
25. 444 and 888 Hz
26. 1; $(0.003) \times (2080) \times [-\sin(2080t)]$
27. No; yes
28. 2π m
29. $1/\Delta\ell; y_m$
30. 1 cm
31. For $n = 1, 5, 9, \ldots;\ A_n = +8y/\pi^2 n^2$.
    For $n = 3, 7, 11, \ldots;\ A_n = -8y/\pi^2 n^2$.
32. 1 m^{-1}
33. About 8×10^{-5} W
34. ± 520 m/s, which is greater than the speed of sound in air
35. The medium (the air) is stationary in one person's frame but moving in the other's.
36. Over 120 million dollars!
37. 1 m/s
38. 79 dB

39. I is too big by about 15 or 20 dB, so about 97% to 99% of the energy must be lost to heat!

40. $f_1 - f_2$

Princeton University
Physics 103
Fall Term 2004

Learning Guide No. 11: Thermodynamics I

Suggested Reading: Tipler, Chapters 17 and 18 (Optional: Section 18-9)

Problem I

A 1-g block of ice is placed in a closed container holding 30 g of air. The initial temperature of the system is $-25°$C. Heat is added at the constant rate of 1 cal/s until the temperature of the system is 125°C.

1. How much of the heat has been used to:

   a) Raise the temperature of the air *alone* from $-25°$C to 125°C? The specific heat of air at constant volume is 0.17 cal/g·°C. **Key 21**

   b) Raise the temperature of the ice alone from $-25°$C to 0°C? The specific heat of ice is very nearly 0.5 cal/g·°C. **Key 32**

   c) Melt the 0°C ice? You can find the numerical data for this part and parts (d) and (e) in Tables 18-1 and 18-2 of Tipler. **Key 1**

   d) Raise the temperature of the water from 0°C to 100°C? **Key 24**

   e) Boil the 100°C water? Assume that the air pressure is exactly 1 atm when the water reaches 100°C so that the water starts to boil. Assume also that all the water boils at exactly 100°C (actually, some won't boil until a higher temperature because of the steam pressure). **Key 17**

   f) Raise the temperature of the steam from 100°C to 125°C? The specific heat of steam near atmospheric pressure is approximately 0.5 cal/g·°C. **Key 20**

2. Make a graph of the temperature of the system as a function of time. Express time in minutes in order to avoid large numbers. **Key 33**

3. If the energy used in parts (1c) through (1e) (turning 0°C ice into 100°C steam) were used instead to lift the gram of ice, how many meters above the earth's surface would the ice be lifted? **Key 10**

Problem II

A 30-g silver teaspoon at room temperature (20°C) is placed into 200 g of hot tea (100°C). No heat escapes the spoon-tea system. You may assume that the specific heat of tea is the same as that of water. The specific heat of silver is 0.0564 cal/g·K. What is the equilibrium temperature of the system? Helping Questions 1 and 2 will get you going. **Key 2**

Problem III

On a nice summer day the air temperature at the surface of the earth is 300 K. What is the average (rms) speed of the nitrogen molecules? Of the oxygen molecules? Use Helping Question 3 if you're stumped. **Key 5**

Problem IV

The mean free path of air molecules at 0°C and 1 atm is about 2×10^{-5} cm.

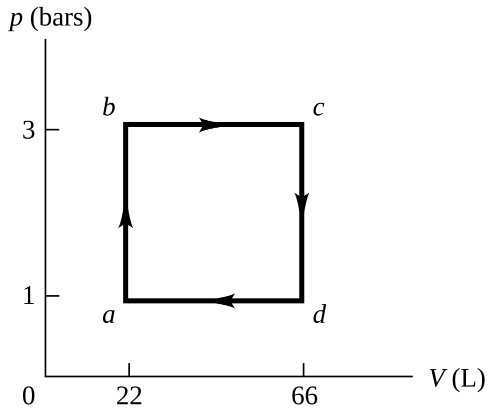
1. What is the effective molecular diameter of an air molecule? **Key 8**
2. At what pressure (at 0°C) would the mean free path be equal to the effective molecular diameter? **Key 3**

Problem V

Work out a semiquantitative explanation of why the earth has an atmosphere but the moon does not. Some useful data: Radius of the earth $R_e = 6.4 \times 10^6$ m, radius of the moon $R_m = 1.7 \times 10^6$ m, local accelerations due to gravity $g_m \simeq g/6$ (remember Neil Armstrong), and daytime temperature of the moon $T_m = 200$°F. Clueless? See Helping Questions 4 and 5. **Key 9**

Problem VI

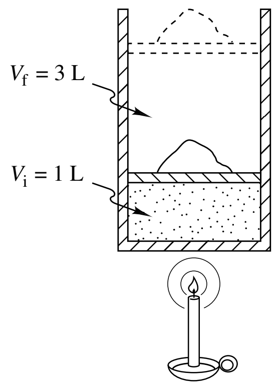
One mole of a nonideal gas is put through the four-stage process as indicated in the accompanying p-V diagram: It begins at roughly atmospheric conditions (a); heat is added at constant volume, causing an increase in pressure ($a \to b$); more heat is added and the gas expands at just the right rate to keep the pressure constant ($b \to c$); and heat is dumped and work is done on the gas as indicated in the diagram in stages $c \to d$ and $d \to a$ to complete the cycle.



1. How much work is done during the cycle? Look at Helping Questions 6 and 7 if you can't figure out the answer. **Key 27**

2. What is the ratio:
$$\frac{\text{(Work done)}}{\text{(Heat taken in)} - \text{(heat dumped)}}$$
for this cycle? Helping Question 8 will remind you of the key idea. **Key 12**

Problem VII



Heat is applied to an ideal diatomic gas, which is maintained at constant pressure by a piston loaded with sand. As heat is added the gas expands from an initial volume $V_\text{i} = 1\,\text{L} = 10^{-3}$ m^3 to a final volume $V_\text{f} = 3$ L. The initial pressure and temperature of the gas are $p_\text{i} = 3$ atm and $T_\text{i} = 0°$C.

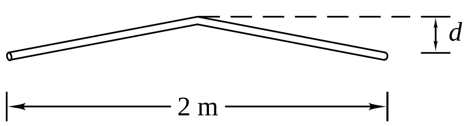
1. How many moles of gas are in the container? Use Helping Question 9. **Key 26**
2. How much work is done by the gas on the piston during the expansion. **Key 11**
3. What is the final temperature of the gas? **Key 16**
4. What heat Q was required? Use Helping Question 10. **Key 18**
5. How much heat went into an increase in the total internal energy of the diatomic gas? **Key 13**
6. How much of the increase in internal energy was used to increase the translational motion of the diatomic gas? **Key 19**
7. Where did the remainder of the increase in internal energy go? **Key 4**
8. Where did the remainder of the heat go? **Key 28**

Problem VIII

A steel rod of length 2 m at 0°C is placed in boiling water and allowed to reach equilibrium.

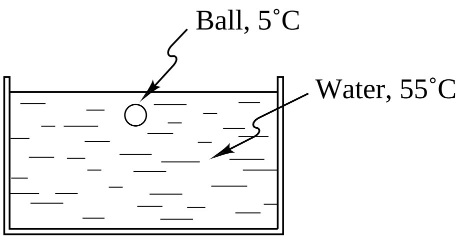
1. What is its new length? You may use Helping Question 11, if your answer doesn't check. **Key 38**

Suppose that the two ends of the rod were constrained to stay exactly 2 m apart. Because of the thermal expansion, the bar bends at its center, as shown in the diagram.



2. By what distance d does the center of the bar rise? Stuck? Look at Helping Question 12. **Key 41**

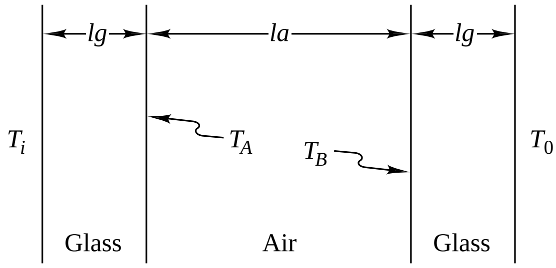
Problem IX



A small ball of radius R and temperature 5°C is observed to be at rest just below the surface of a large pool of water that is maintained at 55°C. Now the ball warms up and comes to thermal equilibrium with the water. What fraction of the volume of the ball is out of the water? (The coefficient of linear expansion of the material of the ball is $\alpha_{\text{ball}} = 10^{-4}/°\text{C}$.) Use Helping Question 13. **Key 34**

Problem X

A double-pane storm window of area A consists of two panes of glass separated by an air space. The thermal conductivity of the air is k_a, that of the glass is k_g. The pane thickness is ℓ_g and the air thickness is ℓ_a.



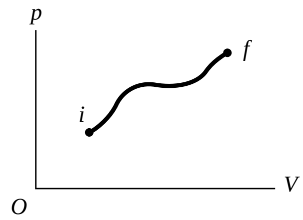
1. If the outside temperature is T_o, which is less than the inside temperature T_i, what is the heat loss per second through the window? **Key 43**

2. What are the temperatures at the inner surfaces of the panes of glass, T_A and T_B? **Key 36**

If you need help, see Helping Questions 14 and 15.

HELPING QUESTIONS

1. What is the relation between the heat lost by the tea and the heat absorbed by the spoon? **Key 14**
2. Can you write an equation for the answer to Helping Question 1 in terms of the initial temperatures T_{spoon} and T_{tea}, the final temperature T, and other constants? **Key 6**
3. What is the relation between average translational kinetic energy and temperature? **Key 31**
4. What is the escape velocity for a particle on the earth's surface? **Key 25**
5. Which distribution gives the probability that a molecule will attain a given speed? **Key 30**
6. In geometrical terms, what is the work done by the process drawn below? **Key 15**



7. In the same geometrical terms as Helping Question 6, what is the work done by the four-stage process in the problem? **Key 7**
8. What does the first law of thermodynamics say? What is the change in internal energy of the gas after one cycle? **Key 23**
9. What does the ideal gas law say? What is atmospheric pressure in $\mathrm{N/m^2}$? **Key 29**
10. What is the molar heat capacity of a diatomic gas at constant pressure? **Key 22**
11. What is the definition of the coefficient of linear expansion α? Use Table 20-1 of Tiplerto find the value of α for steel. **Key 37**
12. To what right triangle can you apply the Pythagorean theorem? **Key 42**
13. What is the relation between the coefficient of volume expansion β and α? **Key 40**
14. When the heat flow has reached a steady state, what is the relation between the heat flow H_i reaching surface A from the inside and the heat flow H_o leaving surface A toward the outside? **Key 39**
15. What are H_i and H_o? **Key 35**

Notes: Temperature, Internal Energy, and the Connection between Them

Most students have more trouble understanding thermodynamics than they have understanding mechanics. If you belong to this group, think about why you're having trouble with thermodynamics. It can't be that you're having more trouble with the *math.* In fact, you need to know less math for thermodynamics than you did for mechanics — there are no vectors in thermodynamics, not even any trigonometry. It must be that you're having more trouble with the *physics.* Everything will fall into place once you understand the three key physical concepts in the suggested reading for this Learning Guide — *temperature, internal energy, and the connection between them.* These three concepts are easiest to understand from the point of view of kinetic theory.

Problem I showed you the necessity of making a distinction between temperature and internal energy. Even though the internal energy of the water-air system was increased at a constant rate, the temperature did not simply increase proportionally. Over certain time intervals the temperature change was proportional to the internal energy change. But the constant of proportionality was different for different intervals; overall, the dependence was rather complicated.

In Problem II you had to use all three concepts to find the equilibrium temperature of the spoon-tea system. Let's look at a formal solution of this problem. Call the final temperatures $T_{\mathrm{f,spoon}}$ and $T_{\mathrm{f,tea}}$. Then if we introduce the extra unknowns $\Delta E_{\mathrm{spoon}}$ and ΔE_{tea}, we have four unknowns, so we need four equations. We get one equation from the zeroth law of thermodynamics, the basic concept of which is temperature:

$$T_{\mathrm{f,spoon}} = T_{\mathrm{f,tea}}.$$

We also get an equation from the first law of thermodynamics, the basic concept of which is internal energy:

$$\Delta E_{\mathrm{spoon}} = -\Delta E_{\mathrm{tea}}.$$

Two more equations come from the definition of specific heat, the basic concept of which is the connection between T and E_{int}:

$$\Delta E_{\mathrm{spoon}} = m_{\mathrm{spoon}} c_{\mathrm{spoon}} (T_{\mathrm{f,spoon}} - T_{\mathrm{spoon}}),$$

and:

$$\Delta E_{\mathrm{tea}} = m_{\mathrm{tea}} c_{\mathrm{tea}} (T_{\mathrm{f,tea}} - T_{\mathrm{tea}}).$$

Let's turn now to kinetic theory and see what light it sheds on these three key concepts. You learned that the temperature of a substance macroscopically at rest is proportional to the average translational kinetic energy of the molecules (this is exactly true for ideal gases and approximately true in other cases). You have to distinguish between temperature and internal energy because of the qualifier "translational kinetic." Why is this type of energy important? When molecules collide they tend to equalize their translational kinetic energies. So if two different types of molecules are in a container, their random collisions serve to even out the average translational kinetic energy but not the average total energy. This is just the content of the zeroth law!

As you know, kinetic theory takes all the mystery out of the first law. The "E_{int}" introduced in Chapter 20 is just the internal energy of the system. For example, you were just using conservation of energy in Problem VII, part (2).

Finally, the connection between temperature and internal energy — heat capacities — is made much less mysterious by kinetic theory. The connection is made by the equipartition of energy. You can get very good estimates for molar heat capacities at constant volume just by counting degrees of freedom. As the text discusses, the molar heat capacities at constant volume for monatomic and diatomic gases are $\frac{3}{2}R$ and $\frac{5}{2}R$, respectively. When the situation is more complicated, you can still use the equipartition theorem qualitatively. Problem I said ice and steam have lower molar heat capacities than water. Can you show that this is reasonable by using the equipartition theorem? Which terms go away? You can get molar heat capacities at constant pressure easily too: For gases, $C_p \simeq C_V + R$, and for solids and liquids, $C_p \simeq C_V$. So if you memorize the values

$$R \simeq 8\frac{\mathrm{J}}{\mathrm{mol}\cdot\mathrm{K}} \simeq 2\frac{\mathrm{cal}}{\mathrm{mol}\cdot\mathrm{K}},$$

you can do many numerical thermodynamics problems quite accurately by using the equipartition theorem instead of tables.

ANSWER KEY

1. 79.5 cal
2. 99.33°C
3. 1000 atm
4. Into rotational motion
5. 517 m/s; 483 m/s
6. $m_{\text{tea}}c_{\text{tea}}(T_{\text{tea}} - T) = m_{\text{spoon}}c_{\text{spoon}}(T - T_{\text{spoon}})$
7. The area of the square
8. 2×10^{-8} cm
9. Compare the quantity

$$P' = \exp\left(-M_{N_2}v_{\text{esc}}^2/2RT\right)$$

for the earth:

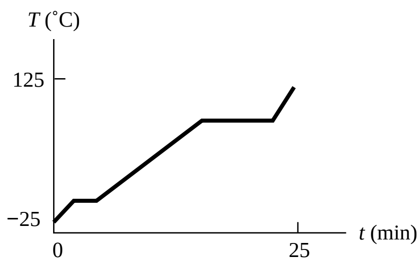
$$\begin{aligned} P' &= \exp\left(-gR_eM_{N_2}/RT\right) \\ &\simeq e^{-774} \\ &\simeq 10^{-336}, \end{aligned}$$

versus the moon:

$$\begin{aligned} P' &= \exp\left(-gR_mM_{N_2}/6RT\right) \\ &\simeq e^{-25} \\ &\simeq 10^{-11}. \end{aligned}$$

Note that 10^{325} is a big number!

10. 3.07×10^5 m
11. 606 J
12. 1
13. 1515 J
14. They are equal.
15. The area under the curve
16. 819 K
17. 539 cal
18. 2121 J
19. 909 J
20. 12.5 cal
21. 765 cal
22. $7R/2$
23. $\Delta E_{\text{int}} = Q - W;\ 0$
24. 100 cal
25. $v_{\text{esc}} = \sqrt{2gR_e}$
26. 0.134 mol
27. 8800 J
28. Into the work done by the gas
29. $PV = nRT$; 1.013×10^5 N/m^2
30. Maxwell's speed distribution (see Tipler, Section 17-5)
31. $E_{\text{av}} = \frac{1}{2}Mv_{\text{rms}}^2 = \frac{3}{2}RT$
32. 12.5 cal
33.



34. 1.5%
35.

$$H_i = \frac{Ak_g(T_i - T_A)}{\ell_g}$$

$$H_o = \frac{Ak_ak_g(T_A - T_o)}{k_g\ell_a + k_a\ell_g}$$

36.

$$T_A = \frac{(k_a\ell_g + k_g\ell_a)T_i + k_a\ell_g T_o}{k_g\ell_a + 2k_a\ell_g}$$
$$T_B = \frac{k_a\ell_g T_i + (k_a\ell_g + k_g\ell_a)T_o}{k_g\ell_a + 2k_a\ell_g}$$

37.

$$\alpha = \frac{1}{\Delta T}\left(\frac{\Delta\ell}{\ell}\right)$$

38. 2.0022 m

39. $H_{\mathrm{i}} = H_o$

40. $\beta \simeq 3\alpha$

41. 4.7 cm

42. The triangle formed by one half the rod, one half the horizontal distance between the ends of the rod, and the vertical rise at the center of the rod.

43.

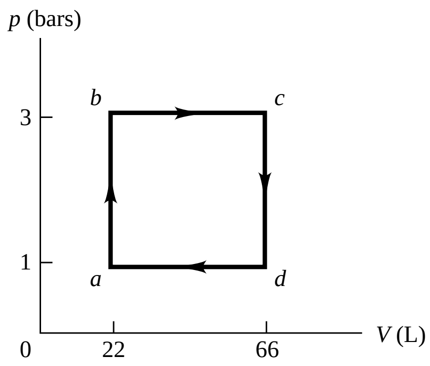
$$H = \frac{A(T_i - T_o)}{(2\,\ell_g/k_g) + (\ell_a/k_a)}$$

Princeton University
Physics 103
Fall Term 2004

Learning Guide No. 12: Thermodynamics II

Suggested Reading: Tipler, Chapter 19

Problem I



The p–V diagram shown is the same as the one used in Learning Guide 11, Problem VI. There you calculated the ratio

$$\frac{\text{(Work done)}}{\text{(Heat taken in)} - \text{(heat dumped)}}$$

in a cycle to be equal to unity, independent of the type of gas used in the engine.

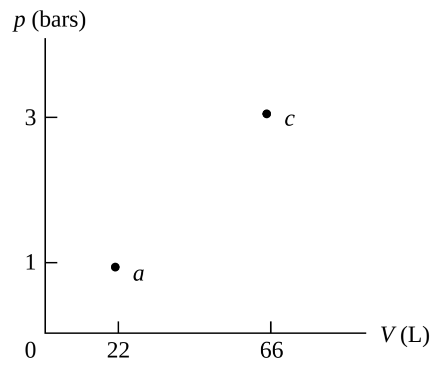
As you know now, a more important ratio is the efficiency of the engine,

$$\eta = \frac{\text{(work done)}}{\text{(heat taken in)}}\,,$$

since the amount of fuel used to produce the work depends on (heat taken in) and not on (heat taken in) − (heat dumped). What is the efficiency of the engine if the working substance of the engine is one mole of ideal monatomic gas? One mole of ideal diatomic gas? If your answer doesn't check after a good try, look at Helping Questions 1, 2, 3, and 4. **Key 7**

Problem II

A gas is put through a Carnot cycle whose graph on a p - V diagram contains the points labeled a and c in Problem I. The gas starts at point a. It is compressed adiabatically until its temperature is the same as the temperature of the gas when it is at point c. It is then expanded isothermally until it reaches point c. It is then expanded adiabatically until its temperature is the same as it was at point a. Finally, it is compressed isothermally until it returns to point a.



Assume that the gas is ideal and monatomic.

1. What is the efficiency of the engine? Stuck? Use Helping Question 5. **Key 8**
2. Indicate the two other corners of the cycle on a p–V diagram. This is somewhat involved. Use Helping Question 6. **Key 17**

Perhaps now you have a better idea of the difficulty of designing a heat engine with near 100% efficiency! Here are two more quick questions to solidify your understanding of heat engines.

3. If the gas used in the Carnot cycle were ideal and diatomic, would the efficiency of the engine be the same as in part (1)? Would the plot of its cycle on a p–V diagram be the same as in part (2)? **Key 20**
4. The cycle of Problem I can be approximated arbitrarily well by Carnot cycles. Why is its efficiency not the same as the efficiencies calculated in parts (1) and (3)? **Key 26**

Problem III

Three thousand calories of heat is added to an ice cube at 0°C turning it into water, also at 0°C.

1. What is the change in entropy, $S_f - S_i$? If your answer doesn't check, reread Section 19-7 of Tipler. **Key 1**

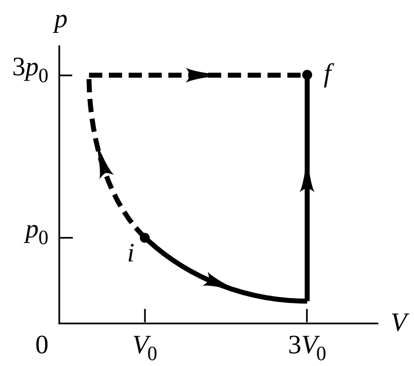
A rock of mass m has a constant specific heat c (that is, the specific heat is independent of temperature). The rock is heated from T_i to T_f.

2. What is the change in entropy $S_f - S_i$? Refer to Helping Questions 7 and 8 as needed. **Key 23**

An ideal gas of n mol undergoes an isothermal expansion from a volume V_i to a volume V_f at temperature T.

3. What is the change in entropy, $S_f - S_i$? If you're stuck after a good effort, use as few of Helping Questions 9, 10, and 11 as possible. **Key 12**

Problem IV

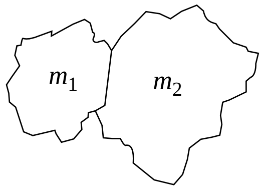


One mole of monatomic ideal gas goes from an initial state i to a final state f by two different processes as indicated in the p–V diagram. Process 1 follows the solid path, whereas process 2 follows the dotted path. The curved line is an isotherm. In terms of p_0, V_0, and the initial temperature T, calculate for each process:

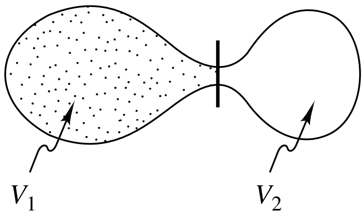
1. The increase in internal energy. See Helping Question 12. **Key 9**
2. The work done by the gas. See Helping Question 13. **Key 18**
3. The heat added to the gas. See Helping Question 14. **Key 13**
4. The increase in entropy. If you're stuck, look at Helping Question 15. **Key 2**

Problem V

The rock and ideal gas of Problem III are each involved in an *irreversible* process. The rock, of mass m_1 and constant specific heat c_1, starts at a temperature T_1. It is placed in contact with a cooler rock of mass m_2, constant specific heat c_2, and temperature T_2. There is a rapid transfer of heat across the boundary, and the rocks reach their equilibrium temperature. No heat is transferred from the rocks to the surroundings, and the rocks do no work.



1. What is the increase in the entropy of the system? Use Helping Questions 16 and 17. **Key 33**



One mole of an ideal gas is in the left chamber of the container sketched here. The valve at the narrow passage is opened, and the gas flows rapidly and turbulently to its equilibrium point. At equilibrium, of course, it is spread evenly throughout the two chambers. Again, no heat is transferred from the gas to its surroundings, and no work is done by the system.

2. What is the increase in entropy of the system? If you're lost, use Helping Questions 18 and 19. **Key 31**

Problem VI

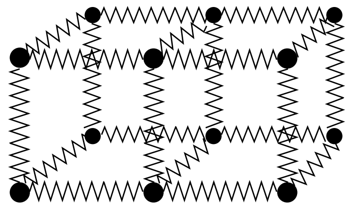
Without looking at the text, *derive* the efficiency of a Carnot cycle working between T_H and T_C. Start from the equation

$$\text{(Entropy taken in)} = \text{(entropy dumped)},$$

and express your answer in terms of T_H and T_C. See Helping Question 20. **Key 3**

Problem VII

The molar heat capacities of metals are *roughly* constant, independent of the type of metal. This is a consequence of the equipartition of energy, discussed in Section 18-5 in the context of gases. Just as hard spheres in random motion form a reasonably good model for monatomic gases, hard spheres in a lattice connected by massless springs form a reasonably good (but not perfect) model for metals. There are 6.02×10^{23} atoms in a mole of metal, so only a small part of the lattice is drawn to the right.



1. If there are N atoms, how many terms are there in the expression for the total energy? Because the lattice is so big, you can neglect edge effects. Use Helping Questions 21 and 22. **Key 16**

2. Use what you know about equipartition of energy to find the molar heat capacity at constant volume of a metal. **Key 36**

The calculation above yields molar heat capacities at *constant volume*. What about the heat capacities for metals at *constant pressure?*. It turns out that this distinction isn't important; for liquids and solids, $C_p \simeq C_V$, because liquids and solids do negligible work when they expand.

3. What is the *molar* heat capacity of water in cal/mol·K? For comparison, also give your answer in terms of R. How does this compare with the molar heat capacities of metals and gases? **Key 24**

HELPING QUESTIONS

1. What are the temperatures T_a, T_b, T_c, and T_d at the corners of the diagram? Along which legs of the cycle is heat taken in? **Key 30**

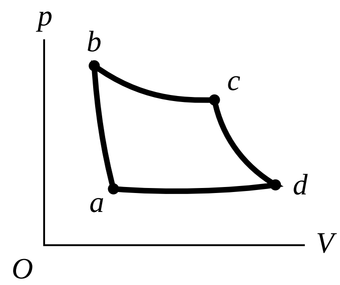
2. What is the molar heat capacity at constant volume for a monatomic ideal gas? For a diatomic ideal gas? Then how much heat is added in $a \to b$ in each of the two cases? **Key 10**

3. What is the molar heat capacity at constant pressure for a monatomic ideal gas? For a diatomic ideal gas? Then how much heat is added in $b \to c$ in each of the two cases? **Key 21**

4. What is the work done in the cycle? **Key 28**

5. What is the efficiency of a Carnot cycle working between temperatures T_H and T_C? **Key 5**

6. The cycle is plotted *very inaccurately.* An exact plot can be made because you can find *equations* describing all four curves. The equation for $b \to c$ is $pV = p_b V_b$. You can get this from the ideal gas law, since $b \to c$ is an *isotherm.* Find equations for the other three curves. Then use these equations to find the two other corners. **Key 37**



7. What is dS in terms of dQ and T? What is the definition of the specific heat c? Combine these two equations into one that doesn't involve dQ. **Key 27**

8. Express the change of entropy $S_\mathrm{f} - S_\mathrm{i}$ as an integral. Can you evaluate this integral by using the equation from Helping Question 7? **Key 14**

9. Express the first law $dE_\mathrm{int} = dQ - dW$ in terms of E_int, T, S, p, and V. Now solve your expression for dS. **Key 11**

10. Since the gas is ideal and the process is isothermal, what is dE_int? **Key 6**

11. Express the change in entropy $S_\mathrm{f} - S_\mathrm{i}$ as an integral. Can you evaluate this integral by using the answers to Helping Questions 9 and 10 and the ideal gas law, $pV = nRT$? **Key 25**

12. What is the formula for the internal energy of a monatomic ideal gas? **Key 34**

13. $dW = ?$ **Key 15**

14. What is the first law of thermodynamics? **Key 22**

15. Use Helping Questions 9 and 12 to find an equation for the change of entropy of an ideal monatomic gas. **Key 29**

16. What is the relation between the heat released by rock 1 and the heat absorbed by rock 2? Can you deduce the equilibrium temperature from this? **Key 4**

17. Use your result from Problem III, part (2).

18. Can you think of a *reversible* process that has the same initial and final states as the irreversible process described? **Key 19**

19. Use your result from Problem III, part (3).

20. Can you express the equation (entropy taken in) = (entropy dumped) in terms of T_H, T_C, Q_H, and Q_C? Now use the definition of efficiency and the first law of thermodynamics. **Key 35**

21. How many terms are there in the expression for the kinetic energy? **Key 38**

22. About how many springs are there? Then how many terms are there in the expression for the potential energy? **Key 32**

Notes: Entropy

In the notes from Learning Guide 11 you saw how the microscopic viewpoint can give you a deeper understanding of the thermodynamic variables T and E_{int} and their associated laws — the zeroth and first laws. In this Learning Guide you'll see how the microscopic viewpoint can give you a deeper understanding of the thermodynamic variable S and its associated law — the second law.

The key to this understanding is interpreting the entropy of a system as the disorder of the system. Let's look at two examples from Problem III. When ice melts, the H_2O molecules go from being regularly arranged in a crystal lattice to being in random motion. Intuitively, the disorder of the system increases, and, sure enough, the entropy change you calculated was positive. When a gas expands, the molecules go from being localized in a relatively small volume to being spread out over a larger volume. Again, intuition says the disorder increases, and again the entropy change you calculated was positive. With this interpretation, the second law says that the disorder of the universe never decreases. So it's not very hard to turn an abstract concept into an intuitive one — just replace the word "entropy" by the word "disorder"!

Entropy is intimately related to heat through the equation $dS = dQ/T$. When you add heat to a system you are increasing its entropy. This makes sense from the point of view of disorder: when you add heat, you are adding energy in a disordered form. But you must be careful to distinguish heat from entropy. As you saw in Problem IV, it doesn't make sense to talk about "heat energy" in a system. It *does* make sense to talk about the entropy or disorder in a system. This is the importance of the innocent-looking T in the equation $dS = dQ/T$. If you add a certain amount of heat dQ to a system, the amount of entropy you add depends on the temperature. Because of this factor T, S is a state variable.

If you take a course in statistical mechanics, you will see that thinking in terms of disorder doesn't just *happen* to give you the right answers; thinking in terms of disorder works because "disorder" really does describe what's physically going on. The basic idea is this: when a system is in equilibrium in a fixed macroscopic state (for a gas, p, V, and T would be fixed), it is microscopically changing state constantly. *Some macroscopic states have more microscopic states corresponding to them than others have.* As an example, consider the gas of Problem V. There are more microscopic states corresponding to the final state (large volume) than there are corresponding to the initial state (small volume). In quantum statistical mechanics, there is actually a number N of possible microscopic states corresponding to a given macroscopic state. The entropy of the macroscopic state is $k \ln N$! Here k is Boltzmann's constant, which is present only for dimensional reasons. So, quite literally, entropy measures disorder!

ANSWER KEY

1. 11 cal/K
2. Process 1:

$$\Delta S = R\ln 3 + \frac{3}{2}R\ln 9 = 4R\ln 3$$

Process 2:

$$\Delta S = -R\ln 3 + \frac{5}{2}R\ln 9 = 4R\ln 3$$

3.

$$\eta_C = 1 - \frac{T_C}{T_H}$$

4. They're equal;

$$T = \frac{c_1 m_1 T_1 + c_2 m_2 T_2}{c_1 m_1 + c_2 m_2}.$$

5.

$$\eta_C = 1 - \frac{T_C}{T_H}$$

6. $dE_{\rm int} = (\text{constant})\,dT = 0$
7. Monatomic: 22%
   Diatomic: 15%
8. 89%
9. Process 1: $12RT$
   Process 2: $12RT$
10. $\frac{3}{2}R$; $\frac{5}{2}R$

$$Q_{\rm mono} = \frac{3}{2}(p_b V_b - p_a V_a);$$
$$Q_{\rm dia} = \frac{5}{2}(p_b V_b - p_a V_a).$$

11.

$$dE_{\rm int} = T\,dS - p\,dV$$
$$dS = \frac{dE_{\rm int}}{T} + \frac{p\,dV}{T}$$

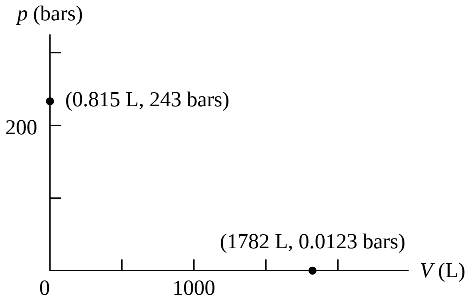
12.

$$\Delta S = nR\ln\frac{V_{\rm f}}{V_{\rm i}}$$

13. Process 1: $RT(12 + \ln 3)$
    Process 2: $RT(20 - \ln 3)$
14.

$$\int_{S_{\rm i}}^{S_{\rm f}} dS; \quad \text{yes}: \quad mc\int_{T_{\rm i}}^{T_{\rm f}} \frac{dT}{T}$$

15. $p\,dV$
16. $6N$
17.



18. Process 1: $RT\ln 3$
    Process 2: $RT(8 - \ln 3)$
19. The gas expands quasi-statically, doing work against a very slowly moving wall. Enough heat is added so that the internal energy remains constant.
20. Yes; no
21. $\frac{5}{2}R$; $\frac{7}{2}R$

$$Q_{\rm mono} = \frac{5}{2}(p_c V_c - p_b V_b)$$
$$Q_{\rm dia} = \frac{7}{2}(p_c V_c - p_b V_b)$$

22. $dE_{\rm int} = dQ - dW$
23.

$$\Delta S = mc\ln\frac{T_{\rm f}}{T_{\rm i}}$$

24. 18 cal/mol·K or about $9R$; besides the translational and rotational kinetic energies of the individual molecules, there is also energy binding the molecules to each other.
25.

$$\int_{S_{\rm i}}^{S_{\rm f}} dS = nR\int_{V_{\rm i}}^{V_{\rm f}} \frac{dV}{V}$$

26. Most of the Carnot cycles are working between temperatures less extreme than T_a and T_c.

27.

$$dS = \frac{dQ}{T}$$

and

$$c = \frac{1}{m}\,\frac{dQ}{dT},$$

so

$$dS = mc\,\frac{dT}{T}.$$

28. $(p_b - p_a)(V_c - V_a)$

29.

$$S_\mathrm{f} - S_\mathrm{i} = nR\,\ln\frac{V_\mathrm{f}}{V_\mathrm{i}} + \frac{3}{2}nR\,\ln\frac{T_\mathrm{f}}{T_\mathrm{i}}$$

30. $T = pV/R$; $T_a = 265$ K; $T_b = 794$ K; $T_c = 2383$ K; $T_d = T_b$. Heat is taken in along the legs $a \to b$ and $b \to c$.

31.

$$R\,\ln\left(\frac{V_1 + V_2}{V_1}\right)$$

32. $3N$; $3N$

33. Entropy increase =

$$m_1c_1\,\ln\frac{T}{T_1} + m_2c_2\,\ln\frac{T}{T_2}$$

where

$$T = \frac{m_1c_1T_1 + m_2c_2T_2}{m_1c_1 + m_2c_2}.$$

34.

$$E_\mathrm{int} = \frac{3}{2}nRT$$

35.

$$\frac{Q_C}{T_C} = \frac{Q_H}{T_H}$$

(entropy is constant along the adiabatic legs).

36. $3R$

37. $a \to b$: $pV^\gamma = p_aV_a^\gamma$, $\gamma = C_p/C_V$
$c \to d$: $pV^\gamma = p_cV_c^\gamma$
$d \to a$: $pV = p_dV_d$

38. $3N$