Physics 305, Fall 2008
Problem Set 2

due Thursday, September 25

1. **Springy 1D Atoms (30 points):** Consider the Hamiltonian $H = H_1 + H_2 + h$ where
$$H_i = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_i^2} + \frac{1}{2}m\omega^2 x_i^2 \ ,$$
and we have added the perturbation
$$h = \epsilon(x_1 - x_2)^2 \ .$$
Let $H_i\phi_n(x_i) = \hbar\omega(n+1/2)\phi_n(x_i)$, $n = 0, 1, 2, \ldots$ where $\phi_n(x)$ are the (properly normalized) one-particle harmonic oscillator wave functions. One can think of this system as a toy model of a two electron atom. Consider the case where the two particles are identical spin 1/2 fermions. Denote the singlet wave function of the two fermions α and the triplet wave function β_i, $i = -1, 0$, or 1.

   a. Ignoring h, express the ground and first excited wave functions of the system in terms of $\phi_n(x)$, α, and β_i. What are the degeneracies of the corresponding two states? What are their energies?

   b. Treating h as a small perturbation, what is the first order correction to the ground state energy? What is the first order correction to the first excited state energy? (If there are degeneracies, you should calculate the first order energy correction for all the states.)

2. **Harmonic Oscillators (15 points):** Consider N identical, one dimensional, non-interacting, spinless particles in a harmonic trap with energy spacing $\epsilon = \hbar\omega$. (As usual, we will take the single particle ground state to have energy $\hbar\omega/2$.) Denote the properly normalized one particle wave functions of the oscillator $\phi_j(x)$ where $j = 1$ is the ground state, $j = 2$ the first excited state and so on. (To make the answer to (a) look nicer, we have re-indexed the $\phi_j(x)$ as compared to the previous problem.)

   a. In terms of the $\phi_j(x)$, what is the ground state wave function for N bosons? N fermions? What is the ground state energy E_B for the bosons? What is the ground state energy E_F for the fermions?

   b. Consider the cases $N = 3$ and $N = 4$. For the bosons, how many states are there with energy $E_B + n\epsilon$, $n = 0, 1, 2, 3, 4, 5$? For the fermions, how many states are there with energy $E_F + n\epsilon$ using the same small values of n? What do you notice about these degeneracies?

3. **Interacting 1D bosons (40 points):** Consider the following Hamiltonian for a system of N identical bosons moving along a line:
$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + g\sum_{i\neq j}\delta(x_i - x_j) \ .$$

This Hamiltonian is an example of an integrable system and can be solved for arbitrary N using the Bethe ansatz. In fact, it has also been realized experimentally by confining cold atoms in cigar shaped traps. We'll content ourselves with studying small values of N.

a. Explain how we can reconstruct the entire wave function from knowledge of the wave function in the region D: $x_1 < x_2 < \cdots < x_N$.

b. Focusing on the region D, we would like to derive boundary conditions on $\psi(x_1, \ldots, x_N)$ when $x_i = x_{i+1}$ and express these conditions purely in terms of ψ in the region D. To that end, verify the following relation:

$$g\,\psi|_{x_i=x_{i+1}} = \frac{\hbar^2}{m} \lim_{\epsilon\to 0^+} \left(\frac{\partial}{\partial x_{i+1}}\psi - \frac{\partial}{\partial x_i}\psi \right)\Bigg|_{x_{i+1}-x_i=\epsilon} .$$

(Comment: There is a factor of two here which I have always found tricky.)

c. Assuming $g < 0$ and $N = 2$, the lowest energy state is similar to the hydrogen atom where the relative motion of the two particles is a bound state but the center of mass motion is a plane wave:

$$\psi_k(x_1, x_2) = \frac{1}{\sqrt{2\pi}} e^{ik(x_1+x_2)/2} \phi(x_2 - x_1) .$$

Find the plane wave normalized wave function $\psi(x_1, x_2)$ where we impose the normalization condition:

$$\delta(k - k') = \int dx_1 dx_2\, \psi^*_{k'}(x_1, x_2)\psi_k(x_1, x_2) .$$

What is the energy of this state?

d. Assuming $g < 0$, find the plane wave normalized ground state wave function $\psi_k(x_1, x_2, x_3)$ for $N = 3$. Here again we know that the center of mass motion will factor out:

$$\psi_k(x_1, x_2, x_3) = \frac{1}{\sqrt{2\pi}} e^{ik(x_1+x_2+x_3)/3} \phi(x_1, x_2, x_3) .$$

What is the energy of $\psi_k(x_1, x_2, x_3)$?