Physics 305, Fall 2008
Problem Set 4

due Thursday, October 9

1. **Band Gaps in 2d metals (20 points):** Consider a square lattice in two dimensions with the crystal potential

$$U(x, y) = -4U \cos(2\pi x/a) \cos(2\pi y/a) \ .$$

Use degenerate first order perturbation theory to find the energy gap at the corner of the Brillouin zone, i.e. for an electron with wave-vector $\vec{k} = (\pi/a, \pi/a)$.

2. **BEC of Lithium-7 (40 points):** At about the same time that Ketterle, Wiemann, and Cornell formed Bose-Einstein condensates of rubidium-87 and sodium-23 atoms (and later won the Nobel prize), Randy Hulet's lab in Texas was trying to form a BEC of lithium-7 atoms in a harmonic trap. We will try to model Hulet's system with a mean-field approach, using the Gross-Pitaevskii equation — essentially a nonlinear generalization of Schrödinger's equation:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2 + \frac{Ng}{2}|\psi|^2$$

where H acts on the wave function ψ. Here ψ is a single particle wave function. At low temperatures, all the N bosons should be in the same state, and the contribution of the interactions to the energy should scale as the local density of the wave function $|\psi|^2$, giving rise to the third term in our Hamiltonian. One very significant difference between these experiments is that lithium-7 atoms attract ($g < 0$) while rubidium-87 and sodium-23 repel ($g > 0$).

   a. Explain how lithium-7 is a boson.

   b. Using a Gaussian trial wave function $\psi = c\, e^{-r^2/2a^2}$, calculate the expectation value of the energy $E(a) = \langle H \rangle$ as a function of a:

$$E(a) = \frac{1}{4}\left(\frac{3\hbar^2}{ma^2} + 3m\omega^2 a^2 + \frac{\sqrt{2}Ng}{2\pi^{3/2}a^3}\right) \ .$$

   c. In the limit in which the interaction energy is large compared to the kinetic energy, minimize $E(a)$ as a function of a for the repulsive case $g > 0$. How do $a_{\min}$ and $\langle H \rangle_{\min}$ scale with N?

   d. Later in the course, I hope to be able to derive the formula

$$g = \frac{4\pi\hbar^2\ell}{m} \ ,$$

where ℓ is the "scattering length" for the bosons. Assume the trap has a frequency $\omega = 2\pi \times 145$ Hz and the "scattering length" $\ell = -1.5$ nm for lithium-7. What is the maximum number of lithium-7 atoms that can be placed in the trap.