Physics 305, Fall 2008
Problem Set 5
due Thursday, October 16

1. 1D molecules (30 points): One could think of this problem as a warm-up for the hydrogen molecule ion presented in Griffiths 7.3. Consider the following model Hamiltonian of a one dimensional molecule:

\[ H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - g\delta(x) - g\delta(x-a), \]

where \( g > 0 \) and \( a > 0 \). In the case where the second delta function is set to zero, the (properly normalized) ground state wave function is

\[ \psi_0(x) = \sqrt{\kappa} e^{-\kappa|x|} \]

where \( \kappa = mg/\hbar^2 \). In the case where the first delta function is set to zero, there is a corresponding ground state wave function we will label \( \psi_a(x) \). Now \( H \) can be solved exactly, but we will instead consider the approximate ground state

\[ \psi(x) = N(\psi_0(x) + \psi_a(x)). \]

a. Calculate the value of \( N \) that makes \( \psi(x) \) properly normalized.

b. Calculate the expectation value of the Hamiltonian \( \langle H \rangle \) in the state \( |\psi\rangle \). Show that you can express the expectation value in the form

\[ \langle H \rangle = -\frac{\hbar^2\kappa^2}{2m} \frac{1 + e^{-\kappa a}(3 + \kappa a) + 2e^{-2\kappa a}}{1 + e^{-\kappa a}(1 + \kappa a)}. \]

c. Imagine the nuclei, represented by the delta functions, experience a repulsive potential of the form

\[ V(a) = \frac{1}{10} \frac{\hbar^2\kappa}{2m a}. \]

At what value of \( \kappa a \) is the total energy minimized? What is the energy at the minimum, expressed in units of \( \hbar^2\kappa^2/2m \)? (You will only be able to determine this value numerically.) What can you conclude from the value of the energy at the minimum?

2. Radioactive Decay (20 points): The two nuclei, \(^{226}\text{Ra}\) and \(^{226}\text{Th}\), with \( Z = 88 \) and \( Z = 90 \), both disintegrate by \( \alpha \) emission with the emitted \( \alpha \) particles having energies of 4.9 MeV and 6.5 MeV, respectively. Assuming that the nuclear radius is the same for both nuclei, \( R = 7.3 \times 10^{-13} \) cm, estimate the ratio of their half-lives using the WKB approximation. The \( \alpha \) particle is less strongly bound as \( Z \) increases (at fixed \( A = 226 \) in this case) because of the Coulomb repulsion among the protons. This suggests the nuclear radius \( R \) actually increases a little with increasing \( Z \). What will be the sign of the effect of this radius change on your ratio of half-lives?
3. **WKB and central force potentials (30 points):** The radial wave functions and eigen energies for bound states of the hydrogen atom with large quantum number [more precisely, large \((n-l)\)] are well approximated using the WKB method. Let’s generalize the Coulomb potential to an attractive power-law central potential with general exponent:

\[
V(r) = -\frac{A}{r^b},
\]

with \(A\) positive and, for parts (a) and (b) below, \(0 < b < 2\). As usual, a particle of mass \(m\) is moving in this potential.

a. Show that within the WKB approximation there are an infinite number of bound states for each value of the angular momentum quantum number \(l\) when \(0 < b < 2\).

b. For the \(l = 0\) states, what are the bound state eigenenergies at large \(n\)? Again, use WKB and its “matching rules.” Express the answer in terms of a finite, dimensionless, unevaluated integral. Look up the integral in a table, or have a computer program such as Mathematica evaluate it for you.