# Physics 305, Fall 2008
# Problem Set 6

due Thursday, November 6

1. **Path integral for a free particle (25 points):** We take a look at the path integral for a free, one dimensional particle of mass $m$.

   a. From class notes, we have
   $$\langle x_f, t_f | x_i, t_i \rangle = \lim_{n\to\infty} \left(\frac{m}{ih\tau}\right)^{(n+1)/2} \int \left(\prod_{j=1}^{n} dx_j\right) \exp\left[\frac{i\tau}{\hbar}\sum_{j=0}^{n}\left(\frac{m}{2}\left(\frac{x_{j+1}-x_j}{\tau}\right)^2\right)\right]$$
   where $t_f - t_i = (n+1)\tau$, $t_f = t_{n+1}$, $t_i = t_0$, $x_f = x_{n+1}$ and $x_i = x_0$. Carry out this path integral.

   b. Check your result from part (a). Let $\psi_k(x) = \langle x|\psi_k\rangle$ be the usual complete, plane wave normalizable basis of states for our free particle. Use this basis to compute $\langle x_f, t_f | x_i, t_i\rangle$ via
   $$\langle x_f, t_f | x_i, t_i \rangle = \int dk \, \langle x_f, t_f|\psi_k\rangle\langle\psi_k|x_i, t_i\rangle \ .$$

   c. (Extra Credit 10 pts) The wave function for a particle at time $t = 0$ is a Gaussian:
   $$\psi(x, t=0) = \langle x, 0|\psi\rangle = Ae^{-x^2/2\sigma^2} \ .$$
   Use the result from part (a) to calculate $\psi(x_f, t_f)$. What is $\langle x^2\rangle$ as a function of time? (Make sure to normalize $\psi(x, t=0)$ properly before beginning.)

2. **Spin precession and "spin echo" (40 points):** The following problem is a warm-up for the time-dependent perturbation theory we will consider after fall break. The Hamiltonian of a $s = 1/2$ spin in a time-dependent magnetic field $\vec{B}(t)$ is
   $$H(t) = \mu\vec{B}(t)\cdot\vec{\sigma} \ ,$$
   where $\mu > 0$ is the spin's magnetic moment and $\vec{\sigma}$ are the Pauli matrices for the spin. Let the spin be initially oriented along the positive $z$-axis at time $t = 0$. We will subject it to a certain sequence of fields that are a caricature of some things that can be done in MRI imaging.

   a. The $\pi/2$ pulse: First apply a field of magnitude $B_x$ along the positive $x$-axis from time $t = 0$ to $t = t_1$. What must the product $B_x t_1$ be in order for this application to cause a 90 degree rotation of the spin?

   b. What is the resulting orientation of the spin at time $t_1$?

   c. Next apply a field of magnitude $B_z$ along the positive $z$-axis, and leave it on for a longer time, so the spin may rotate many times. What is the resulting time dependence of $\mu\langle\vec{\sigma}\rangle$, the expectation value of the spin's magnetic moment at time $t$ (it is a vector)? For a proton spin in a 1 Tesla field, what is the precession frequency?

In solids (e.g. bone, etc.), variation in the material's magnetic properties cause the field $\vec{B}$ to vary slightly over the positions of the various nuclear spins. Variation in the magnetic field in turn causes the spins' precession rates to differ, and the rotating magnetic moments quickly get out of phase, eliminating the coherent macroscopic precessing magnetization that would otherwise be present. The following "spin echo" technique can bring back the coherence:

After $B_z$ is on from time $t_1$ to $t_1 + t_2$, turn it off and apply $B_x$ as in part (a), but now for time $2t_1$, so the spins rotate by 180 degrees about the $x$-axis (a "$\pi$-pulse"). Then apply $B_z$ again for the same time duration $t_2$. Note $t_2$ is long, so each spin turns many times about the $z$-axis during the applications of $B_z$, and the product $B_z t_2$ is not chosen to give any particular number of rotations of the spin. However, we assume the two applications of field along the $z$-axis are identical in magnitude and duration.

d. At the end of these 4 applications of the field ($B_x$ for $t_1$, $B_z$ for $t_2$, $B_x$ for $2t_1$, and then $B_z$ for $t_2$ again), what is the orientation of the spin? Show that it does not depend on $B_z$, so all the spins will be aligned at this time, even though $B_z$ varies over the material. This reappearance of a coherent rotating magnetization is called the "spin echo".

# Have a happy fall break!